THE DIAMETER GAME

JÓZSEF BALOGH, RYAN MARTIN, AND ANDRÁS PLUHÁR

Abstract. A large class of the so-called Positional Games are defined on the complete graph on n vertices. The players, Maker and Breaker, take the edges of the graph in turns, and Maker wins iff his subgraph has a given – usually monotone – property. Here we introduce the d-diameter game, which means that Maker wins iff the diameter of his subgraph is at most d. We investigate the biased version of the game; i.e., when the players may take more than one, and not necessarily the same number of edges, in a turn. The 2-diameter game has the property that Breaker wins the game in which each player chooses one edge per turn, but Maker wins as long as he is permitted to choose 2 edges in each turn whereas Breaker can choose as many as $0.25n^{1/7}/(\ln n)^{3/7}$.

In addition, we investigate d-diameter games for $d > 1$. The diameter games are strongly related to the degree games. Thus, we also provide a generalization of the fair degree game for the biased case.

1. Introduction

A so-called positional game can be viewed, in the most general setting, as a game in which two players – Maker and Breaker – occupy vertices in a hypergraph. Maker wins if he can occupy all vertices of one hyperedge. Breaker wins if he can occupy at least one vertex of each hyperedge. For more information on these games, see the excellent survey included in [4].

At each turn, let Maker make a moves and Breaker make b. In this paper we let the Maker start the game. We call such games $(a : b)$-games. If $a = b$, the game is *fair*, otherwise it is *biased*. If $a = b > 1$, the game is *accelerated*. In [4], Beck asked about the behavior of accelerated versus unaccelerated games. The specific example is one in which players alternately take coins of the same type and place them on a circular table so that the coins do not overlap. The player who, for the first time, cannot make a proper move, loses. In the $(1 : 1)$-game, Maker can win by placing a coin in the center and make moves that reflect the second player's move. Beck observed, however, that it is not clear that Maker has a winning strategy for the

1991 *Mathematics Subject Classification.* 05C65, 91A43, 91A46.

Key words and phrases. Positional games, random graphs, diameter.

The first author's research is partially supported by NSF grant DMS-0302804 and by OTKA grants T034475 and T049398.

The second author's research is partially supported by NSA grant H98230-05-1-0257.

The third author's research is partially supported by OTKA grants T034475 and T049398.

(2 : 2)-game. Beck also observed that the arithmetic progression game $AP(m)$ has the same property. Both questions are open. It is well-studied in the literature that a game may have completely different outcomes if it is played as (1 : 1) or $(a : b)$ game, where either a or b is greater than 1. Two classical results are due to Beck [1] and Chvátal and Erdős [6] (in our paper see Theorems 5 and 6). Other variants of biased games were investigated by Pluhár [8, 9] and Sieben [10].

Here, we investigate the so-called 2-diameter game and in particular we prove that in the (1 : 1)-game, Breaker wins; but, in the (2 : 2)-game, Maker wins.

Let K_n (the complete graph on n vertices) be the "board." Maker wins if some given (usually monotone) graph property $\mathcal{P}$ holds for the subgraph of his edges. An important guide to understanding such games is the so-called *probabilistic intuition*, for more details and examples see [3]. In the probabilistic intuition, we substitute the perfect players with "random players." It means that we consider the random graph $G(n, p)$ with $p = a/(a + b)$; i.e., the edge distribution in the random graph is the same as in the game. It was demonstrated in a number of cases that Maker wins if $\mathcal{P}$ holds, while Breaker wins if $\mathcal{P}$ does not hold for $G(n, p)$ with probability close to one, see e.g., the papers [3, 5, 11]. Here the property $\mathcal{P}_d$ is that the graph has diameter at most d. We denote the corresponding d-diameter game by $\mathcal{D}_d(a : b)$, or more briefly, by $\mathcal{D}_d$ if $a = b = 1$. Note that this is the first case that it is shown that the (1 : 1) and the (2 : 2)-game have a different outcomes, when the **minimum** size of a winning set of both players is $n - 1$.

The first non-trivial case is when $d = 2$. Since the diameter of $G(n, 1/2)$ is 2 with probability close to one, one would expect that Maker wins easily the game $\mathcal{D}_2$. Nevertheless, the probabilistic intuition fails!

Proposition 1. *If $n \leq 3$, then Maker wins the game $\mathcal{D}_2$. If $n \geq 4$, then Breaker has a winning strategy for the the game $\mathcal{D}_2$.*

A little acceleration of the game changes the outcome completely. In the case where $a = 2$, we expect the probabilistic intuition to work; we prove a weaker result.

Theorem 2. *Maker wins the game $\mathcal{D}_2(2 : \frac{1}{4}n^{1/7}/(\log n)^{3/7})$, and Breaker wins the game $\mathcal{D}_2(2 : (2+\epsilon)\sqrt{n/\ln n})$ for any $\epsilon > 0$, provided n is large enough.*

Before dealing with the general case, it is worth discussing the problem $\mathcal{D}_3(1 : b)$. Note that the threshold of the property $\mathcal{P}_3$ is about $n^{-1/3}$; i.e., $G(n, p)$ has property $\mathcal{P}_3$ with probability close to 1 if $p = n^{-1/3+\epsilon}$, and it does not have property $\mathcal{P}_3$ if $p = n^{-1/3-\epsilon}$ for arbitrary $\epsilon > 0$. The game $\mathcal{D}_3(1 : b)$ defies the probabilistic intuition again. However we suspect that the $\mathcal{D}_3(3 : b)$ game again agrees with the probabilistic intuition; i.e., the breaking point should be $b_0 \approx n^{2/3} \times \text{polylog}(n)$ but we do not have the courage to state a conjecture for the breaking point for the $\mathcal{D}_3(2, b)$ game.

Theorem 3. *Maker wins the game $\mathcal{D}_3(1 : c_1\sqrt{n/\ln n})$, and Breaker wins the game $\mathcal{D}_3(1 : c_2\sqrt{n})$, for some $c_1, c_2 > 0$, provided n is big enough.*

Theorem 3 is implied by the following more general theorem.

Theorem 4. *There exists a constant $c_0 > 0$ such that if d is an integer, $3 \le d \le c_0 \ln n / \ln\ln n$, then there is a $c_1 = c_1(d) > 0$, such that Maker wins the game $\mathcal{D}_d(1 : c_1(n/\ln n)^{1-1/\lceil d/2 \rceil})$ if n is large enough.*

Furthermore, there is a constant $c_2 > 0$, depending only on a such that if d is an integer, $3 \le d \le c_2 \ln n/(\ln\ln n)$, then there exist $c_3 = c_3(d) > 0$ and $c_4 = c_4(a, d) > 0$ such that Breaker wins the games $\mathcal{D}_d(1 : c_3 n^{1-1/(d-1)})$ and $\mathcal{D}_d(a : c_4 n^{1-1/d})$, provided n is big enough.

Note that this theorem says that Maker achieves diameter $2k$ by achieving diameter $2k - 1$ for any integer $k \ge 2$. We conjecture that the correct break point is close (up to polylog factor) to the "Breaker's" bound.

The rest of the paper is organized as follows: Section 2 describes general theorems and auxiliary games, such as the degree game and expansion game. Section 3 proves the result of the 2-diameter game. Section 4 proves the results of the d-diameter game for $d \ge 3$.

2. Auxiliary games

2.1. **Positional Games.** Recall that formally a Maker-Breaker Positional Game is defined as follows. Given an arbitrary hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, Maker and Breaker take a and b elements of $V(\mathcal{H})$ per turn. Maker wins by taking all elements of at least one edge $A \in E(\mathcal{H})$, otherwise Breaker wins. The celebrated Erdős-Selfridge theorem gives a condition for Breaker's win when $a = b = 1$, see [7]. We recall its general, biased version that is due to József Beck, see [1].

Theorem 5 (Beck [1])**.** *If $\mathcal{F}$ is the family of winning sets of a positional game, then Breaker has a winning strategy in the $(a : b)$ game when*

$$\sum_{A \in \mathcal{F}} (1+b)^{-|A|/a} < 1.$$

Theorem 6 (Chvátal-Erdős [6])**.** *(i) If $\mathcal{F}$ is a c-uniform family of k* **disjoint** *winning sets, then Maker has a winning strategy in the $(b : 1)$-game when*

$$c \le (b-1) \sum_{i=1}^{k-1} \frac{1}{i}.$$

(ii) If $\mathcal{F}$ is a c-uniform family of k **disjoint** *winning sets, then Maker has a winning strategy in the $(b : 2)$-game when*

$$c \le \frac{b-1}{2} \sum_{i=1}^{k-1} \frac{1}{i}.$$

2.2. **Biased degree games.** In proving theorems on the diameter games, we apply some auxiliary degree games. In such games one player tries to distribute his moves uniformly, while the other player's goal is to obtain as many edges incident to some vertex as possible. Given a graph G and a prescribed degree d, Maker and Breaker play an $(a : b)$ game on the edges of G. Maker wins by getting at least d edges incident to each vertex. For $G = K_n$ and $a = b = 1$ this game was investigated thoroughly in [12] and [3]. They showed that Maker wins if $d < n/2 - \sqrt{n \log n}$, and Breaker wins if $d > n/2 - \sqrt{n}/12$.

This is in good agreement with the probabilistic intuition, since in $G_{n,1/2}$ the degrees of all vertices fall into the interval $[n/2 - \sqrt{n \log n}, n/2 + \sqrt{n \log n}]$ with probability close to one. We are mainly interested in the case of $G = K_n$. When $a \neq b$, analogously one would expect that Maker wins if $d < an/(a+b) - c'\sqrt{n \log n}$, and Breaker wins if $d > an/(a+b) - c''\sqrt{n}$ for some $c', c'' > 0$.

Here we are interested in giving conditions for Maker's win only, so this will suffice:

Lemma 7. *Let $a \leq n/(4 \ln n)$ and n be large enough. Then Maker wins the $(a : b)$ degree game on K_n if $d < \frac{a}{a+b} n - \frac{6ab}{(a+b)^{3/2}} \sqrt{n \ln n}$.*

Proof of Lemma 7. We use a little modification of the weight function argument of Beck, see [3]. Consider the hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, where $V(\mathcal{H}) = E(K_n)$. The set of edges $E(\mathcal{H})$ contains the set A_v for each vertex $v \in K_n$, where A_v is the set of edges incident to v.

In the i^{th} step let X_i and Y_i be the set of edges selected by Maker and Breaker, respectively. The (λ_1, λ_2)-weight of a hyperedge A in the i^{th} step is

$$w_i(A) = (1 + \lambda_1)^{|Y_i \cap A| - (bn/(a+b)+k)} (1 - \lambda_2)^{|X_i \cap A| - (an/(a+b)-k)},$$

where $k = \frac{6ab}{(a+b)^{3/2}} \sqrt{n \ln n}$ and the values of λ_1, λ_2 will be specified later. For any (graph) edge e, let

$$w_i(e) = \sum_{e \in A} w_i(A) \qquad \text{and} \qquad T_i = \sum_A w_i(A).$$

We want to ensure three properties:

(1) If Breaker wins in the i^{th} step, then $T_i \geq 1$,
(2) $T_{i+1} \leq T_i$,
(3) $T_0 < 1$.

Property (1) is trivially true. Maker follows the greedy strategy, that is he always chooses the maximum weight edge available. Let w be the weight of the largest weighted edge **before** Maker makes his last a^{th} move. This means that Maker will reduce the value of T_i by at least $a\lambda_2 w$. When Breaker moves, he will add b edges. So, we have an inequality for T_{i+1}:

$$T_{i+1} \leq T_i - a\lambda_2 w + \left((1 + \lambda_1)^b - 1\right) w.$$

To ensure Property (2), we need to have

$$(1+\lambda_1)^b \le 1 + a\lambda_2. \tag{1}$$

In order to ensure Property (3), we require

$$T_0 = n(1+\lambda_1)^{-bn/(a+b)-k}(1-\lambda_2)^{-an/(a+b)+k} < 1.$$

This simplifies to

$$1+\lambda_1 > n^{\frac{a+b}{bn+k(a+b)}}(1-\lambda_2)^{-\frac{an-k(a+b)}{bn+k(a+b)}}. \tag{2}$$

To satisfy (1) and (2), we need

$$n^{\frac{a+b}{bn+k(a+b)}}(1-\lambda_2)^{-\frac{an-k(a+b)}{bn+k(a+b)}} < 1+\lambda_1 \le (1+a\lambda_2)^{1/b}.$$

Hence, we merely need to verify the existence of a $\lambda_2 > 0$ that gives

$$n^{b(a+b)} < (1+a\lambda_2)^{bn+k(a+b)}(1-\lambda_2)^{abn-kb(a+b)}. \tag{3}$$

Let α be the unique negative root of the equation $1+x = \exp\{x-x^2\}$. Note that $1+x \ge \exp\{x-x^2\}$ for all $x \ge \alpha$, and $\alpha \approx -0.684$. Observe, if $(a+b)/a^2 \ge \alpha^2 n/\ln n$, then $an/(a+b) \le \frac{3ab}{(a+b)^{3/2}}\sqrt{n\ln n}$, and we have nothing to prove. Otherwise one may substitute $\lambda_2 = \sqrt{\frac{(a+b)\ln n}{a(a+1)n}} < -\alpha$, and use the lower bound on $1+x$. So it is enough to see that (3) holds as long as

$$\begin{aligned} n^{b(a+b)} &< \exp\left\{\lambda_2 k(a+b)^2 + \lambda_2^2\left((b-a^2)(a+b)k - ab(a+1)n\right)\right\} \\ 2b(a+b)\ln n &< k\left[(a+b)^2\sqrt{\frac{(a+b)\ln n}{a(a+1)n}} + \frac{(a+b)^2\ln n}{a(a+1)n}(b-a^2)\right] \end{aligned}$$

$$\frac{2b\sqrt{a(a+1)}}{(a+b)^{3/2}}\sqrt{n\ln n} < k\left[1+(b-a^2)\sqrt{\frac{\ln n}{a(a+1)(a+b)n}}\right]. \tag{4}$$

Since it is true that

$$(b-a^2)\sqrt{\frac{\ln n}{a(a+1)(a+b)n}} \ge -\sqrt{a\ln n/n} \ge -1/2,$$

then, by the assumption $a \le n/(4\ln n)$ and the fact that $3ab \ge 2b\sqrt{a(a+1)}$, inequality (4) will be satisfied if

$$\frac{6ab\sqrt{n\ln n}}{(a+b)^{3/2}} \le k.$$

□

Lemma 8. *Let $n > 2a$. Breaker wins the $(a : b)$ degree game on K_n if $d > a \left\lfloor \frac{n}{a+b} \right\rfloor$.*

Proof of Lemma 8. In the first round Breaker chooses a vertex, v, which Maker has not touched and chooses all of his edges to be incident to that vertex in every round. At the end of the game, Maker has chosen at most $a \left\lfloor \frac{n-1}{a+b} \right\rfloor$ edges incident to that vertex. □

2.3. **Expansion game.** In the Expansion Game, Maker attempts to ensure that for every pair of disjoint sets R and S, where $|R| = r$ and $|S| = s$, there is an edge between R and S. We may assume that $s \geq r$.

Lemma 9. *Maker wins the Expansion Game with parameters r and s if one of the following holds:*

(a) $2b \ln n < r \ln(a+1)$,
(b) $b \ln n < r \ln(a+1) < 2b \ln n$ *and* $s > \frac{rb \ln n}{r \ln(a+1) - b \ln n}$,
(c) $r \ln(a+1) < b \ln n$ *and* $n - s < \frac{nr \ln(a+1)}{b \ln n + r \ln(a+1)}$.

Proof of Lemma 9. We use Theorem 5. Maker's and Breaker's roles are switched in that lemma and we know that the corresponding hypergraph has $\binom{n}{r}\binom{n-r}{s}$ hyperedges, each of size rs.

First we prove that condition (a) is sufficient; i.e.,

$$\binom{n}{r}\binom{n-r}{s}(1+a)^{-rs/b} \leq \exp\left\{(s+r)\ln n - \frac{rs}{b}\ln(a+1)\right\} \leq \exp\left\{s\left(2\ln n - \frac{r}{b}\ln(a+1)\right)\right\} < 1, \tag{5}$$

because $2 \ln n < r \ln(a+1)/b$, and we can apply Theorem 5 directly. For (b), just plug in the appropriate value of s to (5) and apply Theorem 5 again.

For the rest, we use similar estimates with $\binom{n-r}{n-r-s}$ in place of $\binom{n-r}{s}$. Hence, the proper binomial coefficient expression is

$$\binom{n}{r}\binom{n-r}{n-r-s}(1+a)^{-rs/b} \leq \exp\left\{(n-s)\left(\ln n + \frac{r}{b}\ln(a+1)\right) - \frac{nr}{b}\ln(a+1)\right\} < 1$$

if (c) is satisfied here. □

3. The 2-diameter game

Lemma 7 immediately implies that Maker wins the $\mathcal{D}_2(a : b)$ game if $a > b$ are fixed and n is large enough. (Actually, a need not be fixed, it can be as large as $(n/(144 \ln n))^{1/3}$ for n large enough and, as long as $a > b$, Maker wins.) Maker plays a degree-game with $d = \left\lceil \frac{n-1}{2} \right\rceil$ on K_n. Since Maker can win this game, the graph has the property that if $u \not\sim v$, then $N(u) \cap N(v) \geq n - 2 - 2\left(n - 2 - \left\lceil \frac{n-1}{2} \right\rceil\right)$. The right hand side is 1 if n is odd and 2 if n is even. This implies the diameter is 2.

3.1. **Proof of Proposition 1.** For $n \leq 3$ the statement is obvious. For $n \geq 4$ regardless of whether Maker or Breaker starts the game, Breaker chooses an edge not incident to that chosen by Maker. Let this edge – the one Breaker chooses – be uv. The strategy of Breaker is: if Maker chooses an edge incident to u, say uw, then Breaker chooses wv. Similarly, if Maker chooses vw, then Breaker chooses wu. Otherwise Breaker may take an arbitrary edge. Clearly, at the end of the game, the pair $\{u, v\}$ has distance at least three. □

3.2. **Proof of Theorem 2.** First we prove that Breaker wins the $\mathcal{D}_2(2 : b)$-game if $b = (2+\epsilon)\sqrt{n/\ln n}$. Breaker plays in 2 phases. In Phase I he picks vertex v, and occupies all possible edges incident to it, at least $(n-3)b/(b+2)$ of them. Let $u_1, \ldots, u_t$ be the list of vertices so that Maker occupied the edge vu_i. Note that

$$t \leq n - 1 - (n-3)b/(b+2) \leq (2n+3b)/(b+2).$$

Now Breaker considers the $n-t-1$ disjoint sets of edges, for a vertex $x \notin \{v, u_1, \ldots, u_t\}$, $E_x = \{xv, xu_1, \ldots, xu_t\}$. By Theorem 6 (ii), Breaker can occupy one of these sets, say E_x, forcing v and x at distance at least 3 from each other, if

$$t + 1 \leq \frac{b-1}{2} \ln(n - t - 1),$$

which is satisfied for $b = (2+\epsilon)\sqrt{n/\ln n}$ if n is large enough.

Now we prove that Maker wins the $(2 : b)$ game, if b is not too large. We shall set r, s and c such that the possible value of b is maximized. Our strategy has two phases. The first one requires approximately $2nr(b+2)$ edges, and the second deals with the rest of the $\binom{n}{2}$ edges. Moreover, in the first phase, Maker will play four subgames, each with a different strategy.

Denote $\deg_B(x)$ to be Breaker's degree at x and $\deg_M(x)$ to be Maker's degree at x.

Phase I. There are $2nr$ rounds in this phase. Each of the following games is played in successive rounds. That is, Maker plays game i in round j iff $i \equiv j \pmod 4$. A vertex is called *high* if it achieves $\deg_B(x) \geq cn/b$ before the end of Phase I.

The four games in Phase I will ensure the following:

- **Game 1. Ratio game.** If vertex x is high, then $\frac{\deg_B(x)}{\deg_M(x)} < 3b$.
- **Game 2. Degree game.** For all vertices x, $\deg_M(x) \geq r$.
- **Game 3. Expansion game.** If $|R| \geq r$ and $|S| \geq s$, then there is a Maker's edge between R and S.
- **Game 4. Connecting high vertices.** We ensure that every high vertex will connect to previously-high vertices by a path of length at most 2. This subgame will continue in Phase II.

Game 1 verification. We can view Game 1 as a $(2:4b)$-game. Here, Maker plays the degree game, for which he has a winning strategy, provided $d < n/(1+2b) - \frac{48b}{(2+4b)^{3/2}}\sqrt{n\ln n}$. This follows from Lemma 7. Suppose, at some time $\deg_B(x) \geq cn/b$ but $\deg_M(x) < \deg_B(x)/(3b)$.

In that case, Breaker would have a winning strategy for the degree game because he could take all $4b$ of his edges incident to x. Even if Maker responds by taking both edges in each round incident to x, we see that at the end of this game, Breaker's degree would be at least:

$$\begin{aligned}
&\deg_B(x) + [n-3-(\deg_B(x)+\deg_M(x))]\,\frac{4b}{4b+2} \\
\geq\;& \deg_B(x) + \left[n-3-\deg_B(x)\left(1+\frac{1}{3b}\right)\right]\frac{4b}{4b+2} \\
\geq\;& (n-3)\left(\frac{2b}{2b+1}\right) + \deg_B(x)\left(\frac{1}{6b+3}\right) \\
\geq\;& (n-3)\left(\frac{2b}{2b+1}\right) + \left(\frac{cn}{b}\right)\left(\frac{1}{6b+3}\right). \qquad (6)
\end{aligned}$$

As long as

$$c \geq 36b^{3/2}\sqrt{\frac{\ln n}{n}} \qquad (7)$$

and n is large enough, we have the following:

$$\frac{cn}{b(6b+3)} \;\geq\; \frac{36b^{3/2}}{b(6b+3)}\sqrt{n\ln n} > \frac{48b}{(4b+2)^{3/2}}\sqrt{n\ln n} + 3.$$

This contradicts the fact that Maker can win the degree game because it allows Breaker, from (6), to force Maker's degree of x to be less than $\frac{n}{2b+1} - \frac{48b}{(4b+2)^{3/2}}\sqrt{n\ln n}$.

Game 2 verification. Here, we play a greedy $(2:4b)$-degree game until the end of Phase I in round $2nr$. The reason that we will not have an error term as we might expect is that we will exploit the result from Game 1.

Maker's strategy is to put an edge incident to a vertex whose Maker degree is less than r. If $\deg_B(x) \geq 3br$, then Game 1 ensures that $\deg_M(x) > r$. If $n-1 > (3b+1)r$, which is satisfied if n is large enough and

$$n \geq 4br, \qquad (8)$$

then it must be the case that we can ensure $\deg_M(y) \geq r$ for all y.

Game 3 verification. Here we play a *virtual* $\left(2 : \frac{\binom{n}{2}}{2nr\cdot 4b} - 2\right)$-expansion game on sets of size r and s. That is, for each $4b$ edges that Breaker chooses, Maker also assumes that Breaker also adds a set of $\binom{n}{2}/(2nr \cdot 4b) - 2 - 4b$ edges arbitrarily. In this way, we can apply results about the completed game to our situation, because Phase I is finished in $2nr$ rounds. This ensures not only an edge between each (R, S) pair, but ensures that it occurs in Phase I.

By Lemma 9, for any fixed ϵ, it is sufficient to have:

$$s \geq r \geq 3 \tag{9}$$

and

$$2\left(\frac{n}{16rb}\right)\ln n < r\ln(2+1) \Leftrightarrow \ln n < \frac{8r^2 b \ln 3}{n}. \tag{10}$$

(We use $n/(16rb)$ in place of "b" in the lemma.)

Game 4 verification. Let ℓ be the maximum possible number of high vertices. Breaker occupies $2nrb$ edges by the end of Phase I. Hence,

$$2nrb \geq \ell\left(\frac{cn}{b}\right),$$

implying

$$\ell \leq \frac{2rb^2}{c}.$$

For $i = 0, \ldots, \lfloor \frac{2nr-4}{4\ell} - 1 \rfloor$, in round $4\,(i\ell + t) + 4$, we will work to ensure that the t^{th} high vertex, denoted x_t, will be connected to each of $x_1, \ldots, x_{t-1}$ by a path of length at most 2.

For the games in rounds $\{4\,(i\ell + t - 1) + 4\}_{i \geq 0}$, we play a different $(1 : 4\ell b)$-game. Maker will choose a path of length 2 between x_t and some x_j, $1 \leq j < t$. Breaker can choose individual edges, and choosing an edge from a 2-path he "occupies it."

Maker will play as Breaker in the standard Erdős-Selfridge game in Theorem 5. There are $t-1$ winning sets, the j^{th} one consists of all of the paths of length 2 from x_j to x_t which have no Breaker edges at the time that x_t becomes high. Since Game 1 ensures that, for a high vertex, x_j, it is the case that $\frac{\deg_B(x_j) - 4b}{\deg_M(x_j)} < 3b$ (we subtract $4b$ from $\deg_B(x_j)$ because the effects of Game 1 may be delayed by a round). Hence $\deg_B(x_j) < \frac{3b(n-1)+4b}{3b+1} < \frac{3bn}{3b+1} + \frac{1}{3}$. Moreover, in the set of four rounds when x_t becomes high, $\deg_B(x_t) \leq \frac{cn}{b} + 4b$. As a result, the size of these winning sets is at least

$$n - 1 - \deg_B(x_j) - \deg_B(x_t) > n\left(\frac{1}{3b+1} - \frac{c}{b}\right) - \frac{1}{3} - 1 - 4b.$$

If b grows slowly (actually, $b = O(\sqrt{n})$ is sufficient), this is at least

$$\frac{n}{b}\left(\frac{1}{4} - c\right).$$

This is a linear term for any $c < 1/4$. So, applying Lemma 5, we see that Maker (playing as Breaker) has a winning strategy as long as

$$(t-1)(1+1)^{-\frac{n}{b}(1/4-c)/(4\ell b)} < 1.$$

Since t is bounded by $\ell \leq \frac{2rb^2}{c}$, the winning strategy is sufficient if

$$\frac{2rb^2}{c} 2^{-nc(1-4c)/(32rb^4)} < 1. \tag{11}$$

Phase II. In the odd turns of this phase, we will connect each pair of vertices that does not currently have a path of length 2 between them. Because of Game 2 and Game 3, for every vertex v, there are less than s vertices that do not have a Maker's path of length 2 between them.

Maker will play a game in which players will select paths of length 2 between unconnected vertices. These paths do not have a Breaker's edge so Maker gets the first move. Breaker, on the other hand, chooses $2b$ edges. If Breaker chooses edge xy, then there are at most s paths that he occupies to connect x to other vertices. Similarly for y. As a result, this can be viewed as a $(1 : 4bs)$-game between Maker and Breaker.

We have to compute the size of the winning sets. If x is high and y is not, then when Phase II begins,

$$\deg_B(x) < \frac{3bn}{3b+1} + \frac{1}{3} \qquad \text{and} \qquad \deg_B(y) < \frac{cn}{b} + 4b.$$

As a result, if n is large enough and $c < 1/4$, then there are at least $(1/4 - c)n/b$ available paths of length 2 between x and y. This holds also if neither x nor y is high.

Using Theorem 5, we see that $|\mathcal{F}| \leq ns/2$, Breaker (playing as Maker) makes $4bs$ moves on each term and Maker (playing as Breaker) makes 1. Hence, Maker (playing as Breaker) has a winning strategy if

$$\frac{ns}{2} 2^{-(1-4c)n/(16b^2 s)} < 1. \tag{12}$$

We compile the results of (7), (8), (9), (10), (11) and (12). Let

$$b = \frac{n^{1/7}}{4(\ln n)^{3/7}}, \qquad r = n^{3/7}(\ln n)^{5/7}, \qquad s = n^{3/7}\ln n \qquad \text{and} \qquad c = 1/8.$$

Conditions (7), (8), (9) and (12) are trivially satisfied for these values as long as n is large enough. Conditions (10) and (11) determine the proper coefficients of b and r. Thus, Maker wins the $\mathcal{D}_2(1 : b)$ game for $b = \frac{n^{1/7}}{4(\ln n)^{3/7}}$. □

4. Results on the general d-diameter game

4.1. **Proof of Theorem 4 – Maker case.** Let $N_i(v)$ denote the i^{th} neighborhood of vertex v in the graph induced by Maker's edges and $B_i(v) = \{v\} \cup \bigcup_{j=1}^{i} N_j(v)$. For $i = 1, \ldots, \lceil d/2 \rceil - 1$ and each vertex v, we identify a set $S_i(v) \subset B_i(v)$ of size r_i such that $r_0 = 1$ and

$$
\begin{aligned}
r_1 &= \frac{n}{\lceil d/2 \rceil b}\left(1 - 7\sqrt{\frac{\lceil d/2 \rceil b \ln n}{n}}\right) \\
r_i &= \frac{n r_{i-1} \ln 2}{\lceil d/2 \rceil b \ln n + r_{i-1} \ln 2} - \sum_{j=0}^{i-1} r_j, \qquad i = 2, \ldots, \lceil d/2 \rceil - 1.
\end{aligned}
$$

Set

$$\beta := \left(\frac{2n}{\ln n}\right)^{1/\lceil d/2 \rceil} \quad \text{and} \quad b := \frac{n \ln 2}{\lceil d/2 \rceil \ln n}\beta^{-1}.$$

Claim 1. *There exists a positive constant c so that if $d \le c \ln n / \ln \ln n$, then for $i = 1, \ldots, \lceil d/2 \rceil - 1$,*

$$\left(1 - 6\beta^{-1/2}\right)^i \frac{\ln n}{\ln 2}\beta^i \le r_i \le \frac{\ln n}{\ln 2}\beta^i.$$

Proof of Claim 1. We will prove this by induction on i. Let $i = 1$ and we have already defined

$$
\begin{aligned}
r_1 &= \frac{n}{\lceil d/2 \rceil b}\left(1 - 7\sqrt{\frac{\lceil d/2 \rceil b \ln n}{n}}\right) \\
&= \frac{\ln n}{\ln 2}\beta\left(1 - 7\sqrt{\ln 2/\beta}\right).
\end{aligned}
$$

Both bounds hold trivially.

Let us assume the bounds hold for r_{i-1} whenever $i \in \{2, \ldots, \lceil d/2 \rceil - 1\}$. Now we compute r_i:

$$
\begin{aligned}
r_i &= \frac{n r_{i-1} \ln 2}{\lceil d/2 \rceil b \ln n + r_{i-1} \ln 2} - \sum_{j=0}^{i-1} r_j \\
&= \frac{n r_{i-1} \beta}{n + r_{i-1}\beta} - \sum_{j=0}^{i-1} r_j \\
&= r_{i-1}\beta\left[1 - \frac{1}{\frac{n}{r_{i-1}\beta} + 1} - i\beta^{-1}\right].
\end{aligned}
$$

Clearly, $r_i \le r_{i-1}\beta$. So, by the inductive hypothesis, $r_i \le \frac{\ln n}{\ln 2}\beta^i$. As to the lower bound,

$$r_i \ge r_{i-1}\beta\left[1 - \frac{r_{i-1}\beta}{n} - i\beta^{-1}\right].$$

To bound this,

$$i\beta^{-1} \le (d/2)\left(\frac{n}{\ln n}\right)^{2/d} \le 3\beta^{-1/2},$$

as long as $d \le c\ln n/\ln\ln n$ for some constant c. Also

$$\frac{r_{i-1}\beta}{n} \le \frac{\ln n}{2\ln 2}\beta^i \le \frac{\ln n}{2\ln 2}\beta^{\lceil d/2\rceil - 1} \le 3\beta^{-1/2},$$

as long as $d \le c'\ln n$ for some constant c'. So, we combine these and see that $r_i \ge r_{i-1}\beta\left(1 - 6\beta^{1/2}\right)$. By the inductive hypothesis, $r_i \ge \left(1 - 6\beta^{1/2}\right)^i \frac{\ln n}{\ln 2}\beta^i$ and the proof is finished. □

Maker's strategy will be to play $\lceil d/2\rceil$ games, playing the k^{th} in round ℓ iff $k \equiv \ell \pmod{\lceil d/2\rceil}$.

For $i = 1$, Maker will play the degree game. For $2 \le i \le \lceil d/2\rceil - 1$, he will play the expansion game with case (c). In Game $\lceil d/2\rceil$, if d is odd, he plays the expansion game (a) on sets of size approximately $\frac{\ln n}{\ln 2}(\beta/2)^{\lceil d/2\rceil - 1}$. If d is even, then he will play the expansion game (a) to ensure that $\left|B_{\lceil d/2\rceil}(v)\right| \ge n/2$.

Game 1. Maker plays the $(1 : \lceil d/2\rceil b)$-degree game. Lemma 7 ensures that Maker can guarantee that for each v, there is a set $S_1(v) \subseteq B_1(v)$ for which

$$\begin{aligned}
|S_1(v)| &\ge \frac{n}{1 + \lceil d/2\rceil b}\left(1 - \frac{6\lceil d/2\rceil b}{\sqrt{1 + \lceil d/2\rceil b}}\sqrt{\frac{\ln n}{n}}\right) \\
&\ge \frac{n}{\lceil d/2\rceil b}\left(1 - \frac{1}{1 + \lceil d/2\rceil b} - 6\sqrt{\frac{\lceil d/2\rceil b\ln n}{n}}\right) \\
&\ge \frac{n}{\lceil d/2\rceil b}\left(1 - 7\sqrt{\frac{\lceil d/2\rceil b\ln n}{n}}\right) = r_1,
\end{aligned}$$

as long as $\lceil d/2\rceil b \ge (n/\ln n)^{1/3}$.

Game 2 to Game $\lceil \mathbf{d/2}\rceil - \mathbf{1}$. Maker plays the $(1 : \lceil d/2\rceil b)$-expansion game (c).

Thanks to Lemma 9 (c), Maker can guarantee a set $S_i(v) \subset B_i(v)\setminus\left(\{v\} \cup \bigcup_{j=1}^{i-1} S_j(v)\right)$ of size

$$\begin{aligned} |S_i(v)| &= \frac{nr_{i-1}\ln 2}{\lceil d/2\rceil \left(\frac{n\ln 2}{\lceil d/2\rceil \ln n}\right)\beta^{-1}\ln n + r_{i-1}\ln 2} - \sum_{j=1}^{i-1}|S_j(v)| - 1 \\ &= \frac{nr_{i-1}\ln 2}{n\ln 2\beta^{-1} + r_{i-1}\ln 2} - \sum_{j=0}^{i-1} r_j = r_i. \end{aligned}$$

Game $\lceil \mathbf{d/2}\rceil$. Maker plays the $(1 : \lceil d/2\rceil b)$-expansion game (a). Note that, here, we have $S_{\lceil d/2\rceil-1}$ of size $r_{\lceil d/2\rceil-1} \geq (1-3\beta^{-1/2})\frac{\ln n}{\ln 2}\beta^{\lceil d/2\rceil-1}$. We just use Lemma 9 (a) with $r = s = r_{\lceil d/2\rceil-1}$ in the case where d is odd and $s = \lceil (n+1)/2\rceil$ in the case where d is even. Of course, $a = 1$ and we use $\lceil d/2\rceil b$ instead of b.

To verify that the conditions of the lemma are satisfied:

$$2n\ln 2\beta^{-1} < \left(1 - 3\beta^{-1/2}\right)^{\lceil d/2\rceil-1}\ln n\beta^{\lceil d/2\rceil-1} < r_{\lceil d/2\rceil-1}\ln 2$$

and this directly satisfies the condition in Lemma 9 (a).

4.2. **Proof of Theorem 4 – Breaker case, $a = 1$.** We will show that Breaker can win if $a = 1$, $b \geq 4d^{1/(d-1)}n^{1-1/(d-1)}$ and n is sufficiently large.

In the first move, Breaker will choose an edge between u and v. (If Maker goes first, we make sure that neither u nor v is incident to Maker's first edge.) At each move, Breaker will play two roles. He will use $d^{1/(d-1)}n^{1-1/(d-1)}$ edges to ensure the maximum Maker degree is small and then will use the remaining edges to ensure that every vertex in the i^{th} Maker's neighborhood of u is adjacent by a Breaker edge to every vertex in the j^{th} Maker's neighborhood of v for $j = 0, \ldots, d-1-i$.

If Breaker devotes $b_1 = d^{1/(d-1)}n^{1-1/(d-1)}$ edges to the maximum-degree game, Lemma 7 (Maker and Breaker switch roles and d is replaced with $n-1-d$) says that Breaker has a winning strategy to keep the maximum degree, Δ, at most

$$\begin{aligned} \Delta &\leq \frac{n}{b_1+1} + \frac{6b_1}{(b_1+1)^{3/2}}\sqrt{n\ln n} - 1 \\ &\leq \frac{n}{b_1} + 6\sqrt{\frac{n\ln n}{b_1}} \\ &\leq \left(\frac{n}{d}\right)^{1/(d-1)} + 6d^{-1/(2d-2)}\sqrt{n^{1/(d-1)}\ln n} \\ &\leq \left(\frac{n}{d}\right)^{1/(d-1)} \exp\left\{6d^{1/(2d-2)}\sqrt{\frac{\ln n}{n^{1/(d-1)}}}\right\}. \end{aligned} \tag{13}$$

The number of edges used next is b_2. We will guarantee that, for all $k \in \{0, \ldots, d\}$, $B_k(u) \cap B_{d-k}(v) = \emptyset$. Let xy be the edge chosen by Maker in the preceding move.

First, we will consider the distance from u:

(1) Let x be at least as close as y to u; i.e., let $i = \mathrm{dist}(x, u) \leq \mathrm{dist}(y, u)$, where dist is the length of the shortest path in the graph induced by Maker's edges. If $i \leq d-2$, then Breaker will choose each edge in $\left(N_k(y), \bigcup_{\ell=0}^{d-i-2-k} N_\ell(v)\right)$, for $k = 0, \ldots, d-i-2$; and each edge in $\left(N_k(y), \bigcup_{\ell=0}^{i-k} N_\ell(u)\right)$, for $k = 0, \ldots, i-1$.

(2) Let $j = \min\{\mathrm{dist}(x, v), \mathrm{dist}(y, v)\}$. There are two possibilities:
- If $j = \mathrm{dist}(y, v) \leq \mathrm{dist}(x, v)$, then Breaker will choose each edge in $\left(N_k(x), \bigcup_{\ell=0}^{d-j-2-k} N_\ell(u)\right)$, for $k = 0, \ldots, d-j-2$; and each edge in $\left(N_k(x), \bigcup_{\ell=0}^{j-k} N_\ell(v)\right)$, for $k = 0, \ldots, j-1$.
- Otherwise, let $j = \mathrm{dist}(x, v) < \mathrm{dist}(y, v)$, then Breaker will choose each edge in $\left(N_k(y), \bigcup_{\ell=0}^{d-j-2-k} N_\ell(u)\right)$, for $k = 0, \ldots, d-j-2$; and each edge in $\left(N_k(y), \bigcup_{\ell=0}^{j-k} N_\ell(v)\right)$, for $k = 0, \ldots, j-1$.

This procedure will ensure that if there is some edge between $N_i(u)$ and $N_j(v)$ that $i + j > d - 1$.

We will, without loss of generality, focus on item (1). The total number of edges needed is at most:

$$\begin{aligned}
&\sum_{k=0}^{d-i-2} \Delta^k \sum_{\ell=0}^{d-i-2-k} \Delta^\ell + \sum_{k=0}^{i-1} \Delta^k \sum_{\ell=0}^{i-1-k} \Delta^\ell \\
&= \frac{1}{\Delta - 1} \sum_{k=0}^{d-i-2} \left(\Delta^{d-i-1} - \Delta^k\right) + \frac{1}{\Delta - 1} \sum_{k=0}^{i-1} (\Delta_i - \Delta_k) \\
&= \frac{1}{(\Delta - 1)^2} \left[(d-i-1)\Delta^{d-i} - (d-i)\Delta^{d-i-1} + 1\right] \\
&\quad + \frac{1}{(\Delta - 1)^2} \left[i\Delta^{i+1} - (i+1)\Delta^i + 1\right].
\end{aligned} \tag{14}$$

Claim 2. *Let Δ, d be positive integers. Then for $i = 0, \ldots, d-1$,*

$$\begin{aligned}
f(i) &:= \left[(d-i-1)\Delta^{d-i} - (d-i)\Delta^{d-i-1} + 1\right] + \left[i\Delta^{i+1} - (i+1)\Delta^i + 1\right] \\
&\leq (d-1)\Delta^d - d\Delta^{d-1} + 1.
\end{aligned}$$

Proof of Claim 2. By symmetry, $f(i) = f(d-1-i)$. So it is sufficient to show that $f(i) \geq f(i+1)$ for $i \in \{0, \ldots, \lfloor (d-3)/2 \rfloor\}$.

$$\begin{aligned}
&f(i) - f(i+1) \\
&\quad = \Big[(d-i-1)\Delta^{d-i} - (d-i)\Delta^{d-i-1} + 1\Big] + \Big[i\Delta^{i+1} - (i+1)\Delta^{i} + 1\Big] \\
&\qquad - \Big[(d-i-2)\Delta^{d-i-1} - (d-i-1)\Delta^{d-i-2} + 1\Big] - \Big[(i+1)\Delta^{i+2} - (i+2)\Delta^{i+1} + 1\Big] \\
&\quad = \Delta^{d-i-2}\Big[(d-i-1)\Delta^2 - (2d-2i-2)\Delta + (d-i-1)\Big] \\
&\qquad - \Delta^{i}\Big[(i+1)\Delta^2 - (2i+2)\Delta + (i+1)\Big] \\
&\quad = (\Delta-1)^2\Delta^{i}(d-i-1)\left[\Delta^{d-2(i+1)} - \frac{i+1}{d-(i+1)}\right].
\end{aligned}$$

Since $i \leq \lfloor (d-3)/2 \rfloor$,

$$\Delta^{d-2(i+1)} \geq \Delta \geq 1 > \frac{i+1}{d-(i+1)}.$$

Thus, it must be the case that $f(i)$ attains its maximum at $i = 0$ or $i = d-1$ and the claim is proven. □

We bound the term (14) as follows:

$$\frac{1}{(\Delta-1)^2}\Big[(d-1)\Delta^{d} - d\Delta^{d-1} + 1\Big] \leq d\Delta^{d-2}\exp\left\{\frac{2}{\Delta}\right\}.$$

This number of edges is sufficient to guarantee (1) and by symmetry is enough to guarantee either condition of (2). We bound the sum as follows:

$$\begin{aligned}
2d\Delta^{d-2}\exp\left\{\frac{2}{\Delta}\right\} &\leq 2d\left(\frac{n}{d}\right)^{\frac{d-2}{d-1}}\exp\left\{6d^{\frac{2d-1}{2d-2}}\sqrt{\frac{\ln n}{n^{1/(d-1)}}} + 2\left(\frac{d}{n}\right)^{\frac{1}{d-1}}\right\} \\
&\leq 2d^{1/(d-1)}n^{1-1/(d-1)}\exp\left\{3d^{\frac{2d-1}{2d-2}}\sqrt{\frac{\ln n}{n^{1/(d-1)}}}\right\}.
\end{aligned}$$

As long as $d \leq \frac{\ln n}{2\ln\ln n}$ and n is large enough, we have that

$$2d^{1/(d-1)}n^{1-1/(d-1)}\exp\left\{3d^{\frac{2d-1}{2d-2}}\sqrt{\frac{\ln n}{n^{1/(d-1)}}}\right\} \leq 3d^{1/(d-1)}n^{1-1/(d-1)}.$$

Hence, we only need $b_1 + b_2 \geq 4d^{1/(d-1)}n^{1-1/(d-1)}$ in order to ensure a Breaker win.

4.3. **Proof of Theorem 4 – Breaker case,** $a \geq 2$**.** For the sake of simplicity, we deal only with the case $a = 2$; the general case is very similar. We will show that Breaker can win if $b \geq 4n^{1-1/d}$ and will do so by simply playing the degree game. As we have seen in (13), Breaker can ensure that

$$\begin{aligned}\Delta &\leq \frac{2n}{b} + 12\sqrt{\frac{n \ln n}{b}} \\ &\leq \frac{n}{4n^{1-1/d}} + 12\sqrt{\frac{n \ln n}{4n^{1-1/d}}} \\ &\leq n^{1/d}\left(\frac{1}{2} + 6\sqrt{n^{-1/d}\ln n}\right) \\ &\leq (2/3)n^{1/d},\end{aligned}$$

as long as n is large enough.

With Δ being the maximum degree, for any vertex v,

$$\begin{aligned}|B_d(v)| &\leq 1 + \sum_{i=0}^{d-1} \Delta(\Delta - 1)^i \\ &= 1 + \Delta\frac{(\Delta-1)^d - 1}{(\Delta-1)-1} \\ &\leq 1 + \left(1 + \frac{2}{\Delta - 2}\right)\left(\Delta^d - 1\right) < 2\Delta^d,\end{aligned}$$

as long as $\Delta \geq 4$.

But then, using the upper bound for Δ and recalling $d \geq 2$,

$$|B_d(v)| < 2\Delta^d < 2\left(\frac{2}{3}n^{1/d}\right)^d \leq \frac{8}{9}n.$$

So, for any vertex v, there is at least one vertex of distance greater than d from it. Note that in order for Breaker to win this degree game, we need to have that $b \leq n/(4\ln n)$. Since $b = 4n^{1-1/d}$, this holds for $d \leq \ln n/(2\ln\ln n)$ and n large enough. □

References

[1] J. Beck, Remarks on positional games. *Acta Math Acad Sci Hungar* **40** (1982), 65–71.

[2] J. Beck, Random graphs and positional games on the complete graph. Random graphs '83 (Poznań, 1983), 7–13, *North-Holland Math. Stud.,* **118**, North-Holland, Amsterdam, 1985.

[3] J. Beck, Deterministic graph games and a probabilistic intuition. *Combinatorics, Probability and Computing* **3** (1994), 13–26.

[4] J. Beck, Positional Games. *Combinatorics, Probability and Computing* **14** (2005), 649–696.

[5] M. Bednarska, and T. Łuczak, Biased positional games and the phase transition. *Random Structures & Algorithms* **18** (2001), 141–152.

[6] V. Chvátal and P. Erdős, Biased positional games. Algorithmic aspects of combinatorics (Conf., Vancouver Island, B.C., 1976). *Ann. Discrete Mathematics* **2** (1978), 221–229.
[7] P. Erdős and J. L. Selfridge, On a combinatorial game. *J. Combinatorial Theory Series A* **14** (1973), 298–301.
[8] A. Pluhár, Generalized Harary Games. *Acta Cybernetica* **13** (1997) 77–83.
[9] A. Pluhár, The accelerated k-in-a-row game. *Theoretical Computer Science* **270** (2002), 865–875.
[10] N. Sieben, Hexagonal polyomino weak $(1, 2)$-achievement games. *Acta Cybernetica* **16** (2004), 579–585.
[11] T. Szabó and M. Stojaković, Positional games on random graphs. *Random Structures & Algorithms* **26** (2005), no. 1-2, 204–223.
[12] L. A. Székely, On two concepts of discrepancy in a class of combinatorial games. Finite and Infinite Sets, Colloq. Math. Soc. János Bolyai, Vol. 37, North-Holland, 1984, 679–683.

Department of Mathematical Sciences, University of Illinois
E-mail address: jobal@math.uiuc.edu

Department of Mathematics, Iowa State University
E-mail address: rymartin@iastate.edu

University of Szeged, Department of Computer Science
E-mail address: pluhar@inf.u-szeged.hu