FIRST-FIT CHROMATIC NUMBER OF PLANAR AND RANDOM GRAPHS

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ABSTRACT. The *First-Fit chromatic number* of a graph is the number of colors needed in the worst case of a greedy coloring. It is also called the *Grundy number*, which is defined to be the maximum number of classes in an ordered partition of the vertex set of a graph G into independent sets $V_1, V_2, \ldots, V_k$ so that for each $1 \leq i < j \leq k$, and for each $x \in V_j$ there exists a $y \in V_i$ such that x and y are adjacent.

In this paper, we study the First-Fit chromatic number of outerplanar and planar graphs, random graphs, and Cartesian products of graphs. We give asymptotically tight results for outerplanar and random graphs. The results on Cartesian products of graphs allow us to generalize some previous results.

1. INTRODUCTION

Given an ordering of the vertices of a simple graph G, a greedy (proper) coloring of G assigns to each vertex the first available color not used on a neighbor vertex earlier in the order. The First-Fit chromatic number is the number of colors needed in a worst-case greedy coloring. An equivalent definition of First-Fit chromatic number is given below.

The *First-Fit chromatic number* of G, written as $\chi_{FF}(G)$, is defined to be the maximum number of classes in an ordered partition of the vertex set of G into independent sets $V_1, V_2, \ldots, V_k$ so that for each $1 \leq i < j \leq k$, and for each $x \in V_j$ there exists a $y \in V_i$ such that x and y are adjacent. A partition with this property is called a *First-Fit partition* or simply *FF-partition*.

Historically, First-Fit partitions and the First-Fit chromatic number are also called Grundy colorings and the Grundy number respectively. The study of Grundy coloring dates back to the 1930's [10], when Grundy used them in the study of kernels of directed graphs. Many researchers have studied this coloring under different names, see [6] for details. It is believed that Christen and Selkow [3] were the first to define and study the Grundy number as a graph parameter. Some recent results about it can be found for example in Füredi, Gyárfás, Sárközy and S. Selkow [7] and Zaker [17].

Date: 16th October 2006.

Research of the first author is supported in part by NSF grants DMS-0302804, DMS-0600303 and DMS-0603769, UIUC Campus Research Board #06139, and OTKA 049398.

Stemming from a natural connection with the classical Dynamic Storage Allocation problem in computer science, Woodall [16] studied the First-Fit chromatic number of an interval graph I_k with maximum clique size k and showed $\chi_{FF}(I_k) = O(k \log k)$. Kierstead [12] eliminated the logarithmic term and showed that $\chi_{FF}(I_k) \leq 40k$. After a series of papers by Kierstead and Qin [13], and Pemmaraju, Raman and Varadarajan [15], Brightwell, Kierstead and Trotter [2] showed that $\chi_{FF}(I_k) \leq 8k$. Kierstead and Trotter [14] improved the lower bound by showing that for every $\varepsilon > 0$, $\chi_{FF}(I_k) > (5-\varepsilon)k$ when k is sufficiently large. We study the First-Fit chromatic numbers of more general families of graphs.

Theorem 1. *Let $\mathcal{F}$ be a family of graphs closed under taking subgraphs. Suppose that the size of graphs in $\mathcal{F}$ is bounded linearly in the order; that is, there exists constants d and s, $d \geq 1$ and $-2d \leq s \leq 0$, such that $e(G) \leq dn(G) + s$ for every $G \in \mathcal{F}$. Then for every $G \in \mathcal{F}$,*

$$\chi_{FF}(G) \leq \log_{\frac{d+1}{d}}(n) - \log_{\frac{d+1}{d}}(d+1) + (2d+2).$$

Irani [11] proved a similar result, showing that for the graphs above, $\chi_{FF}(G) = O(d \log n)$. However, our constants are slightly better and the proof techniques are different.

As a corollary, we obtain upper bounds of the First-Fit chromatic numbers of planar and outerplanar graphs.

Theorem 2. *(i) Let G be an n-vertex planar graph. Then*

$$\chi_{FF}(G) \leq \log_{4/3}(n) + 8 - \log_{4/3}(4).$$

(ii) If G is additionally outerplanar, then

$$\chi_{FF}(G) \leq \log_{3/2}(n) + 4 - \log_{3/2}(3).$$

For infinite many integers n we construct planar graphs on n vertices with First-Fit chromatic number $\log_{6^{1/6}} n$ (note that $6^{1/6} \approx 1.348$), and we show that the upper bound for outerplanar graphs is asymptotically best possible.

We also study the First-Fit chromatic number of random graphs and give an asymptotically tight result.

Theorem 3. *Let $G = G_{n,1/2}$ be a random graph with edge density $1/2$. Then with high probability (as $n \to \infty$),*

$$\left(1 - \frac{36}{\sqrt{\ln n}}\right)\frac{n}{\log_2 n} \leq \chi_{FF}(G) \leq \left(1 + \frac{10\log_2\log_2 n}{\log_2 n}\right)\frac{n}{\log_2 n}.$$

Note that the chromatic number of the random graph $G_{n,1/2}$ is asymptotically $n/(2\log_2 n)$ almost surely (see [1]), and the ratio of the two chromatic numbers is asymptotically 2. We also obtain a tight concentration result of the First-Fit chromatic number of the random graph, which is analogous to the result of Shamir and Spencer (see [1]) about the tight concentration of the chromatic number of the random graph.

We discuss the First-Fit chromatic number of Cartesian products of graphs in the last section. Effantin and Kheddouci [4] proved that if G is bipartite, then $\chi_{FF}(G\Box P_3) \geq \chi_{FF}(G)+2$. Using this, they proved that $\chi_{FF}(C_{m_1}\Box C_{m_2}\Box \ldots \Box C_{m_k}) = 2k+1$, where $k \geq 2$, $m_i \geq 4$ and m_i is even (the case when all cycles are C_4 requires additional work). Note that this result gives a short proof for the First-Fit chromatic number of hypercubes which was originally obtained by Hoffman and Johnson [9]: A k-dimensional hypercube Q_k has FF-chromatic number $k+1$ except that $\chi_{FF}(Q_2) = 2$.

Here we generalize the above result to include odd cycles and prove the same lower bound for $G\Box P_3$ and $G\Box P_4$, but allow G to be non-bipartite as long as $\chi_{FF}(G)$ is even. As a result, $\chi_{FF}(C_{m_1}\Box C_{m_2}\Box \ldots \Box C_{m_k}) = 2k+1$ for all $k \geq 2$ and $m_i \geq 4$. For contrast, we constructed graphs G with the property that $\chi_{FF}(G\Box P_2)$ is as large as $2\chi_{FF}(G)$.

This paper is organized as follows: in Section 2, we show the results on planar graphs, in Section 3, we discuss the upper bound and lower bounds of random graphs respectively, and in Section 4, we prove the results on products of graphs.

2. First-Fit Chromatic Number of planar Graphs

We begin with a technical, purely combinatorial lemma about sequence of integers. The reader might want to skip the proof at the first time of reading.

Lemma 4. *Given a sequence $a_1, a_2, \ldots, a_t$ of integers and two constants $d \geq 1$ and $s \in [-2d, 0]$ which satisfy*

(i) $a_i \geq 1$ *for* $i = 1, \ldots, t$,

(ii) $\sum_{\ell=1}^{t} a_\ell = n$,

(iii) $\sum_{\ell=i}^{t} (\ell - i)a_\ell \leq d\left(\sum_{\ell=i}^{t} a_\ell\right) + s$, *for* $i = 1, \ldots, t-d$.

Then

$$t \leq \log_{\frac{d+1}{d}}(n) - \log_{\frac{d+1}{d}}(d+1) + (2d+2).$$

Proof. Given a sequence $a_1, a_2, \ldots, a_t$ that satisfies the hypotheses, we define a "shift" operation S_j on such a sequence as follows:

$$S_j(a_1, a_2, \ldots, a_t) = a_1, \ldots, a_{j-1}, a_j + 1, a_{j+1} - 1, a_{j+2}, \ldots, a_t.$$

Clearly, if $a_1, a_2, \ldots, a_t$ satisfies (i) and (ii), then $S_j(a_1, a_2, \ldots, a_t)$ does as well if $a_{j+1} \geq 2$.

For each $j = t-1, \ldots, 1$, we repeatedly apply S_j to $a_1, a_2 \ldots, a_t$ while $a_{j+1} > 1$ and (iii) holds. After this process we obtain a new sequence $b_1, b_2, \ldots, b_t$ that satisfies the conditions of the lemma for the same d and s, but, for each $j = 1, \ldots, t-1$, either $b_{j+1} = 1$ or $b_{j+1} \geq 2$ and (iii) does not hold for $S_j(b_1, b_2, \ldots, b_t)$.

Let i_0 be the smallest index j such that

(a) for any i with $j < i \le t$, $b_i = 1$, and

(b) for any i with $1 \le i \le j$, $b_i \ge 2$ and $S_{i-1}(b_1, b_2, \dots, b_t)$ violates (iii).

Claim A. $t - (2d+1) \le i_0 \le t - (2d-2)$.

Proof. Note that if $b_1, b_2, \dots, b_t$ satisfies (iii), then $S_j(b_1, b_2, \dots, b_t)$ satisfies (iii) for $i \ne j+1$. Thus, if $S_j(b_1, b_2, \dots, b_t)$ violates (iii), then the violation occurs for $i = j+1$:

$$\sum_{\ell=j+1}^{t} (\ell - j - 1) b_\ell \ge d \left(\sum_{\ell=j+1}^{t} b_\ell \right) - d + s + 1. \tag{1}$$

Since $S_{i_0-1}(b_1, b_2, \dots, b_t)$ does not satisfy (iii), by (1), we have $\left(\frac{t-i_0+1}{2} - d\right)(t-i_0) - s \ge d+1$, where we use the fact that $b_i = 1$ for all $i_0 < i \le t$.

On the other hand, (iii) where $i = i_0$ implies that

$$\sum_{\ell=i_0}^{t} (\ell - i_0) b_\ell \le d \left(\sum_{\ell=i_0}^{t} b_\ell \right) + s, \text{ that is, } \left(\frac{t-i_0+1}{2} - d \right)(t-i_0) - s \le d b_{i_0}.$$

Thus, if $\left(\frac{t-i_0+1}{2} - d\right)(t-i_0) - s \ge d+1$, then $b_{i_0} > 1$. Therefore,

$$\left(\frac{t-i_0+1}{2} - d \right)(t-i_0) - s \ge d+1 \text{ if and only if } b_{i_0} > 1. \tag{2}$$

Define

$$f(x) = \left(\frac{x+1}{2} - d \right) x - s = \frac{1}{2}x^2 - (d - 1/2)x - s.$$

By (iii), we may always apply S_j for $j \ge t-d$ as long as $a_j \ge 2$, thus we have $i_0 \le t-d+1$, and so $t - i_0 \ge d-1$. Note that $f(x)$ is increasing on the interval $[d-1/2, \infty)$.

First we have $i_0 \le t-(2d-2)$. In fact, if $d \ge 3$ for $x \le 2d-3$, $f(x) \le f(2d-3) = -2d+3-s \le 3 < d+1$. Thus by (2), $b_{t-x} \le 1$ for every $x \le 2d-3$. Therefore $i_0 < t-(2d-3)$, that is, $i_0 \le t-(2d-2)$ if $d \ge 3$. It is clear that for $d \le 2$, $i_0 \le t-1 \le t-(2d-2)$ as well.

We also have $i_0 \ge t-(2d+1)$. In fact, since $f(2d+1) = 2d+1-s \ge 2d+1 > d+1$, by (2) $b_{t-(2d+1)} > 1$. By the definition of i_0, $i_0 \ge t-(2d+1)$. □

Claim B. $b_i \ge 2$ for all $i \le i_0$, and as a consequence, $S_{i-1}(b_1, b_2, \dots, b_t)$ violates (iii).

Proof. We use downward induction on the index i. The base case is when $i = i_0$ and we are done.

Thus, we assume the inductive hypothesis is true for i, where $1 < i \le i_0$. Hence

$$\sum_{\ell=i}^{t} (\ell - i) b_\ell \ge d \left(\sum_{\ell=i}^{t} b_\ell \right) - d + s + 1. \tag{3}$$

(iii) for $i-1$ states that

$$\sum_{\ell=i-1}^{t} (\ell - i + 1) b_\ell \leq d \left(\sum_{\ell=i-1}^{t} b_\ell \right) + s. \tag{4}$$

Subtracting (4) from (3), we have

$$\sum_{\ell=i}^{t} b_\ell \leq d b_{i-1} + d - 1, \tag{5}$$

which implies

$$\begin{aligned} db_{i-1} \geq \sum_{\ell=i}^{t} b_\ell + 1 - d &\geq (t - i + 2) + 1 - d \qquad \text{since } b_i \geq 2, \\ &\geq (t - (d-3)) - i_0 \\ &\geq d + 1 \qquad\qquad \text{since } i_0 \leq t - (2d-2). \end{aligned}$$

Hence $b_{i-1} \geq 2$, and additionally $S_j(b_1, b_2, \ldots, b_t)$ violates (iii) when $j = i-2$. This completes the induction. □

By Claims A and B, for $b_1, b_2, \ldots, b_t$ we have that (iii) holds and $b_i = 1$ for $i > i_0$ and $b_i \geq 2$ for $i \leq i_0$.
For $i \leq i_0$, we also have from (1) that:

$$d \left(\sum_{\ell=i}^{t} b_\ell \right) - d + s + 1 \leq \sum_{\ell=i}^{t} (\ell - i) b_\ell \leq d \left(\sum_{\ell=i}^{t} b_\ell \right) + s.$$

We know from (5) that for $i = 1, \ldots, i_0 - 1$,

$$db_i \geq \sum_{\ell=i+1}^{t} b_\ell + 1 - d.$$

Let $S_k := \sum_{\ell=k}^{t} b_l$. Then for $k \leq i_0 - 1$,

$$d\left(S_k - (d-1)\right) \geq (d+1)\left(S_{k+1} - (d-1)\right). \tag{6}$$

Thus, iterating (6) on k, we have (note that $S_1 = n$)

$$n - (d-1) = S_1 - (d-1) \geq \left(\frac{d+1}{d} \right)^{i_0 - 1} \left(S_{i_0} - (d-1)\right).$$

Taking logarithms of both sides and solving for i_0,

$$i_0 \leq \log_{\frac{d+1}{d}} (n - (d-1)) - \log_{\frac{d+1}{d}} (S_{i_0} - (d-1)) + 1.$$

Note that for $i_0 \le t-(2d-2)$ we have $S_{i_0} = t - i_0 + b_{i_0} \ge 2d$, and that for $i_0 \ge t-(2d+1)$ we have

$$\begin{aligned} t \le i_0 + 2d + 1 &\le \log_{\frac{d+1}{d}}(n-(d-1)) - \log_{\frac{d+1}{d}}(d+1) + 1 + (2d+1) \\ &\le \log_{\frac{d+1}{d}}(n) - \log_{\frac{d+1}{d}}(d+1) + (2d+2). \end{aligned}$$

□

Now we are prepared to prove Theorems 1 and 2.

Proof of Theorem 1. Suppose that $\chi_{\mathrm{FF}}(G) = t$. Then there exists a First-Fit partition $\langle V_1, V_2, \ldots, V_t \rangle$ of $V(G)$.

For $1 \le i \le t$, let G_i be the subgraph $G[V_i, V_{i+1}, \ldots, V_t]$ induced by the last $t-i+1$ parts. By definition of $\langle V_1, V_2, \ldots, V_t \rangle$,

$$e(G_i) \ge \sum_{\ell=i}^{t} (\ell - i) \left| V_\ell \right|.$$

Since G_i is a member of $\mathcal{F}$, G_i satisfies the edge bound

$$e(G_i) \le d \left(\sum_{\ell=i}^{t} |V_\ell| \right) + s.$$

Hence, the sequence $|V_1|, |V_2|, \ldots, |V_t|$ satisfies the conditions of Lemma 4, and so

$$\chi_{\mathrm{FF}}(G) \le \log_{\frac{d+1}{d}}(n) - \log_{\frac{d+1}{d}}(d+1) + (2d+2).$$

□

Proof of Theorem 2. (i) If G is a planar graph with $n \ge 3$ vertices, then $e(G) \le 3n-6$. Since planarity is preserved when taking subgraphs, the result then follows by applying Theorem 1.
(ii) Since outerplanar graphs with n vertices have at most $2n-4$ edges, we similarly obtain the result. □

Proposition 5. *For every n_0, there is an $n > n_0$ such that there exists an n-vertex planar graph with First-Fit chromatic number at least $\log_{6^{1/6}} n$.*

Proof. A planar graph is called a *triangulation* if all of its faces are triangles. There exists a triangulation H (see Figure 1) with 18 vertices that has First-Fit chromatic number (at least) 9 (the vertices with label i belong to the U_i in an FF-partition $\cup_{i=1}^{9} U_i$ of H). The graph H has the additional property that the vertices of H can be covered by exactly 6 independent triangles.

Given a triangulation G_1 and a set $\mathcal{F}$ of triangles that cover every vertex of G_1, we can substitute a copy of H for every triangle of $\mathcal{F}$. This forms a new triangulation G_2 where $n(G_2) = 6n(G_1)$ and $\chi_{FF}(G_2) \ge \chi_{FF}(G_1) + 6$ (indeed, $V(G_2) = \bigcup_{i=1}^{t+6} U_i$ with $U_i = V_{i-6}$ when $i \ge 7$ gives an FF-partition of G_2, where $\bigcup_{j=1}^{t} V_i$ is an optimal FF-partition of G_1). Note that G_2 has a set of $6|\mathcal{F}|$ triangles that cover every vertex of G_1.

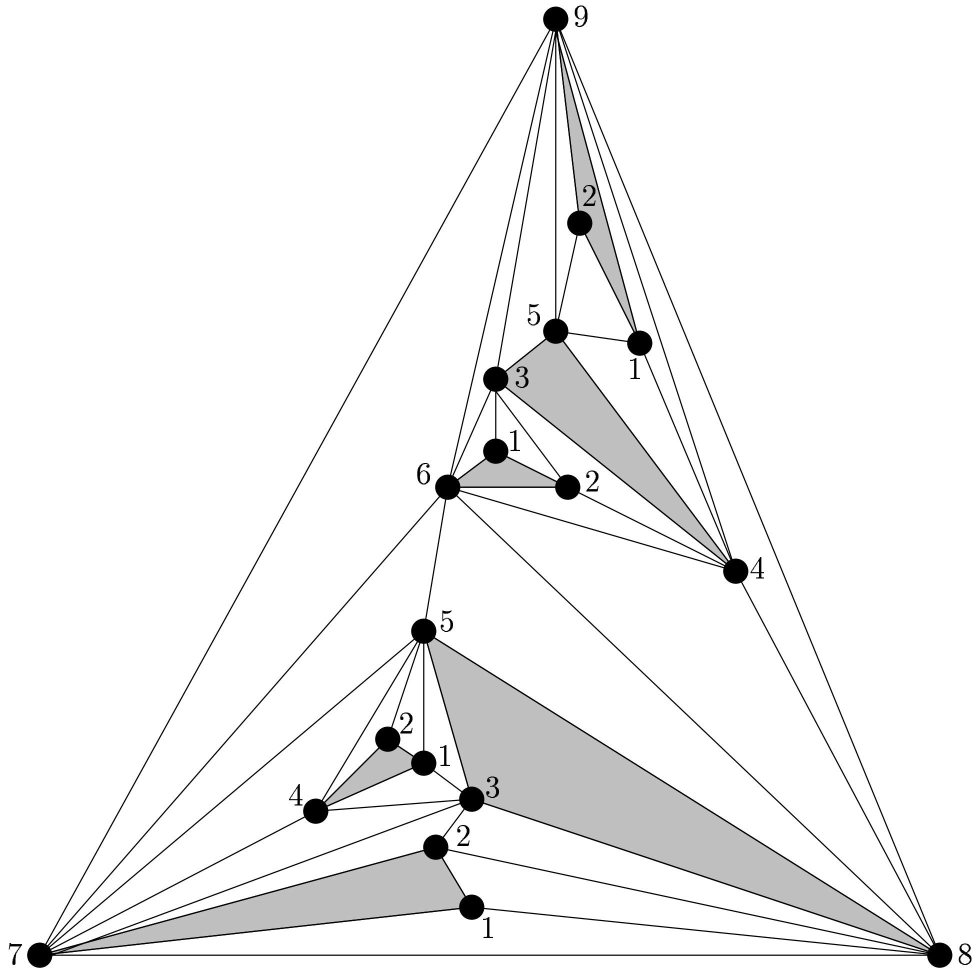

FIGURE 1. An 18-vertex graph H with First-Fit chromatic number 9.

By iterating this procedure starting with $G_1 = C_3$, we obtain a sequence G_i of graphs with $3 \cdot 6^{i-1}$ vertices and First-Fit chromatic number at least $3 + 6(i-1)$. Hence, we have constructed a series of planar graphs whose asymptotic First-Fit chromatic number is at least $\log_{6^{1/6}} n$, where n is the number of the vertices of the graph. □

By considering some more complicated graphs H, we could get a better lower bound, but we do not believe this method can show that the upper bound is tight.

The following construction gives a lower bound for the First-Fit chromatic number of outerplanar graphs.

Proposition 6. *For every n_0, there is an $n > n_0$ such that there exists an n-vertex outerplanar graph with First-Fit chromatic number at least* $\log_{\frac{3}{2}}(n+2) - \log_{\frac{3}{2}}(5) + 3$.

Proof. We construct a sequence of outerplanar graphs as follows:

(1) $G_0 = C_3$.

(2) Suppose the outer face of G_i is $v_1 v_2 \dots v_{n_i}$ in order, where n_i is the number of vertices of G_i. Generate G_{i+1} by adding $k = \lceil n_i/2 \rceil$ vertices $u_1, \dots, u_k$ such that u_j is adjacent to v_{2j-1} and v_{2j}, where $j = 1, \dots, k$ (if n_i is odd, then u_k is adjacent to v_{n_i} and v_1).

Then G_{i+1} is also outerplanar and $\chi_{FF}(G_{i+1}) \geq \chi_{FF}(G_i) + 1$. As $\chi_{FF}(G_0) = 3$, we have $\chi_{FF}(G_i) \geq i + 3$. Since $n_i = n_{i-1} + \lceil n_{i-1}/2 \rceil \leq 3n_{i-1}/2 + 1$ and $n_0 = 3$, we have $n_i \leq 5(3/2)^i - 2$. Thus $\chi_{FF}(G_i) \geq i + 3 \geq \log_{\frac{3}{2}}(n_i + 2) - \log_{\frac{3}{2}}(5) + 3$. □

So the upper bound given by Theorem 2 (ii) for outerplanar graphs is asymptotically sharp.

3. The First-Fit chromatic number of random graphs

Proof of Theorem 3. Recall that $G_{n,p}$ denotes the random graph on n labeled vertices in which every edge is chosen randomly and independently with probability p. Let $\varepsilon = \varepsilon(n) = \frac{10 \log_2 \log_2 n}{\log_2 n}$. Since n is large, for the sake of clarity of exposition, we will ignore the issue of integrality of quantities.

Upper Bound. Let $\ell = (1 + \varepsilon)\frac{n}{\log_2 n}$. A k-partition $\langle V_1, \dots, V_k \rangle$ of $V(G)$ is *good* if for every $j \geq 2$, every vertex in V_j has a neighbor in V_i for each $i < j$. Note that a good partition, as its classes are not necessary independent sets, may not be an FF-partition, but an FF-partition is always a good partition.

Let $k = \chi_{FF}(G)$ and assume that $k \geq \ell$. If G has a good k-partition, then it also has a good ℓ-partition by combining parts.

For an ℓ-partition P of $V(G)$, define

$$f(G, P) = \begin{cases} 1, & \text{if } P \text{ is a good } \ell\text{-partition of } G; \\ 0, & \text{otherwise.} \end{cases}$$

Then the number of good ℓ-partitions of G is $\sum_P f(G, P)$. Therefore, the expected number of good ℓ-partitions in G is

$$s := \mathbb{E}\left[\sum_P f(G_{n,1/2}, P)\right] = \sum_P \mathbb{E}[f(G_{n,1/2}, P)] = \sum_P \mathbb{P}r(P \text{ is a good } \ell\text{-partition of } G).$$

Let $p(P)$ be the probability that P is a good ℓ-partition for G.

Claim C. For every partition P, $p(P) < 2^{-n \log_2 n}$.

Proof. For $G = G_{n,1/2}$, consider an ℓ-partition $P = \langle V_1, \dots, V_k \rangle$ of $V(G)$, with $|V_i| = t_i$. Define $T_i = (1 - 2^{-t_i})^{n - \sum_{j \leq i} t_j}$. Then it follows from the definition of good partitions that

$$p(P) \leq \prod_{i=1}^{\ell-1} (1 - 2^{-t_i})^{n - \sum_{j \leq i} t_j} = \prod_{i=1}^{\ell-1} T_i.$$

We claim that

$$\prod_{i=1}^{\ell-1} T_i < 2^{-n\log_2 n}.$$

It is clear that $T_i \le 1$ for all i. If

$$t_i \le (1-\varepsilon/2)\log_2 n \tag{7}$$

and

$$n - \sum_{j\le i} t_j = \sum_{j>i} t_j > \frac{n}{\log_2^2 n}, \tag{8}$$

then we have a sharper estimate on T_i : $\quad T_i \le (1-2^{-(1-\varepsilon/2)\log_2 n})^{n/\log_2^2 n}$.

We observe that there are at least $\frac{\varepsilon n}{8\log_2 n}$ many i's that satisfy (7) and (8). Indeed, there are at most $\frac{n}{(1-\varepsilon/2)\log_2 n}$ many i's violating (7), thus there are at least $\ell - \frac{n}{(1-\varepsilon/2)\log_2 n} \ge \frac{\varepsilon n}{4\log_2 n}$ many i's satisfying (7).

On the other hand, for every i such that $\ell - i \ge \frac{n}{\log_2^2 n}$ we have $\sum_{j>i} t_j > \frac{n}{\log_2^2 n}$, that is, $n - \sum_{j\le i} t_j > \frac{n}{\log_2^2 n}$. Note if $i \le (1+7/8\varepsilon)\frac{n}{\log_2 n}$ then $\ell - i \ge \frac{n}{\log_2^2 n}$. So there are at least $(1+7/8\varepsilon)\frac{n}{\log_2 n}$ many i's satisfying (8). Hence there are at least

$$\ell - \left(\ell - \frac{\varepsilon n}{4\log_2 n}\right) - \left(\ell - (1+7/8\varepsilon)\frac{n}{\log_2 n}\right) \ge \frac{\varepsilon n}{8\log_2 n}$$

many i's satisfy both (7) and (8) (for $\log_2\log_2 n > 8$).
Therefore,

$$\begin{aligned} p(P) &\le \left[\left(1-2^{-(1-\varepsilon/2)\log_2 n}\right)^{\frac{n}{\log_2^2 n}}\right]^{\varepsilon n/(8\log_2 n)} = \left[1-n^{-(1-\varepsilon/2)}\right]^{\varepsilon n^2/(8\log_2^3 n)} \\ &\le \exp\left(-n^{-(1-\varepsilon/2)}\frac{\varepsilon n^2}{8\log_2^3 n}\right) = \exp\left(-n^{1+\varepsilon/2}\,\frac{\varepsilon}{8\log_2^3 n}\right) \le \exp(-n\log_2 n), \end{aligned}$$

where in the last inequality we used the fact that $\varepsilon n^{\varepsilon/2} > 8\log_2^4 n$ for our choice of ε and sufficiently large n. □

Since there are at most $n!2^n$ ordered partitions of an n-vertex set, with Claim C, we have $s < n!\, 2^n\, 2^{-n\log_2 n} \sim 2^n\, (n/e)^n\sqrt{2\pi n}\, 2^{-n\log_2 n} = (2/e)^n\sqrt{2\pi n} \to 0$. Thus with high probability, there are no good k-partitions when $k \ge \ell$. So $\chi_{FF}(G) < \ell$, and we completed the proof of the upper bound.

Lower Bound. For a vertex x in a graph G, let $\overline{N}(x)$ be the set of non-neighbors of x in G. Let $\overline{N}(A) = \cap_{x\in A}\overline{N}(x) - A$. In our proof the key statement is Lemma 8, whose proof based on repeated use of Chernoff's inequality.

Lemma 7. *[Chernoff bound; see [1] p 263] Let $G = G_{n,1/2}$ be a random graph, T a subset of $V(G)$ of size t, and x a vertex in $V(G) - T$. Then*

$$\mathbb{P}r\left[\left|\overline{N}(x) \cap T\right| > \left(\frac{1}{2} + \gamma\right) t\right] < e^{-2\gamma^2 t}.$$

Let $\delta = 18/\sqrt{\log_2 n}$.

Lemma 8. *Let $G = G_{n,1/2}$ be a random graph. Then for large n, G contains a maximal independent set C of size at most $(1+\delta)\log_2 n$ with probability at least $1 - n^{-2}$. Note that during the process of choosing C there is no information used on the graph spanned by $V(G) - C$.*

Proof. We construct a maximal independent set S iteratively in three phases. We start with $S^0 = \emptyset$. In the first phase we extend this set, in each step by adding a vertex v_i, having $S^{i+1} = S^i \cup \{v_i\}$.

Phase I. This phase lasts for r iterations, until $\overline{N}[S^r] > (\ln n)^3$.
Let $\alpha := \alpha(n) = 1/\sqrt{\ln n}$. Let x_1 be an arbitrarily chosen vertex of G. We pick a sequence of vertices $x_2, x_3, \ldots, x_r$ inductively as follows: let $A_i = \bigcap_{j=1}^{i} \overline{N}[x_j]$. While $|A_i| > (\ln n)^3$, arbitrarily choose x_{i+1} in A_i.

Note that the selection of the vertices $x_1, x_2, \ldots, x_i$ is independent of the edges in the subgraph of G induced by A_i. Hence, the subgraph induced by A_i is a random graph with distribution $G_{|A_i|,\frac{1}{2}}$, and we expect that x_{i+1} is adjacent to about half of the vertices in A_i. By Lemma 7, and using that $|A_i| > (\ln n)^3$,

$$\mathbb{P}r\left[|\overline{N}(x_{i+1}) \cap A_i| > (1/2+\alpha)|A_i|\right] < e^{-2\alpha^2|A_i|} < e^{-2\alpha^2(\ln n)^3} < e^{-8\ln n} = n^{-8}. \tag{9}$$

Thus,

$$\mathbb{P}r\left[|\overline{N}(x_{i+1}) \cap A_i| \le (1/2+\alpha)|A_i| \text{ for all } 1 \le i < r\right] > 1 - rn^{-8} > 1 - n^{-7}.$$

As $|A_0| = n$ and for every $i \le r$ we have $|A_{i+1}| \le (1/2+\alpha)|A_i|$, we can conclude that

$$r \le \log_{\frac{1}{\frac{1}{2}+\alpha}} n < \log_2 n\left[1 + \frac{5}{\ln 2}\frac{1}{\sqrt{\log_2 n}}\right] < \log_2 n\left[1 + \frac{8}{\sqrt{\log_2 n}}\right]. \tag{10}$$

The middle inequality follows from the inequalities $1 - x/2 > \frac{1}{1+x} > 1 - x$ and $\ln(1-x) > -2x$.

Phase II. In the second phase we have the same process, but because of $|\overline{N}[S^r]| = o(n)$, we shall have (9) type of estimate only after adding $m := \ln\ln n$ vertices, at each iteration, to the independent set. We proceed with this phase until $180\frac{\ln n}{\ln\ln n} \le \overline{N}[S^i]$.

We will add m independent vertices $y_1^i, y_2^i, \ldots, y_m^i$ to S^i, one at a time.

Subphase A(i): Addition of the i^{th} group of m vertices to S^{i-1}. Let

$$S^i(j) := S^{i-1} \cup \{y_1^i, y_2^i, \ldots, y_j^i\}.$$

As before, pick y_j^i arbitrarily in $B^i(j-1) := \overline{N}[S^i(j-1)]$, (note that $S^i(0) = S^{i-1}$). While $j \leq m$ and $|B^i(j-1)| > \frac{180 \ln n}{\ln \ln n}$, arbitrarily choose y_j^i in $B^i(j-1)$. To summarize, addition of $\{y_1^i, y_2^i, \ldots, y_m^i\}$ to S^{i-1} forms S^i.

By Lemma 7, and using the fact that $|B^i(j-1)| > \frac{180 \ln n}{\ln \ln n}$,

$$\mathbb{P}r\left[\left|B^i(j)\right| > \frac{2}{3}|B^i(j-1)|\right] < \exp\left[-\frac{1}{18}|B^i(j-1)|\right] \leq \exp\left[-10\frac{\ln n}{\ln \ln n}\right].$$

These events are independent for different pairs of i and j. Thus, the probability that each pair (i,j) has more than $\frac{2}{3}|B^i(j-1)|$ non-neighbors in $B^i(j-1)$ is

$$\mathbb{P}r\left[\left|\overline{N}(y_j^i) \cap B^i(j-1)\right| > \frac{2}{3}|B^i(j-1)| \text{ for all } 1 \leq i \leq m\right] < \left(e^{-\frac{10 \ln n}{\ln \ln n}}\right)^m = n^{-10}.$$

Since $|\overline{N}[S^i]| > \frac{2}{3}|\overline{N}[S^{i-1}]|$ implies that $\left|\overline{N}(y_j^i) \cap B^i(j-1)\right| > \frac{2}{3}|B^i(j-1)|$ for all $1 \leq j \leq m$, we have

$$\mathbb{P}r\left[|\overline{N}[S^i]| \leq \frac{2}{3}|\overline{N}[S^{i-1}]|\right] > 1 - n^{-10}. \tag{11}$$

We repeat Subphase A(i) while $\overline{N}[S^i] \geq \frac{180 \ln n}{\ln \ln n}$. If in the last iteration of Subphase A(i) we cannot choose m vertices, then we simply add the vertices chosen so far to S. Suppose that Subphase A is repeated q times. By (11), $q \leq \log_{\frac{2}{3}}\left((\ln n)^3\right) + m = O(\ln \ln n)$. The "$+m$" comes from the last possibly-incomplete iteration of Subphase A. Hence, $O\left((\ln \ln n)^2\right)$ vertices are added to S during Phase II.

Phase III. $\overline{N}[S] \leq \frac{180 \ln n}{\ln \ln n}$.

We choose a maximal independent set C in $\overline{N}[S]$ to add to S in the same greedy way as in Phase I. As $\overline{N}[S]$ spans a random graph with edge-density $1/2$, it does not have an independent set of size $10\sqrt{\log_2 n}$ with probability at least

$$1 - \binom{|\overline{N}[S]|}{10\sqrt{\log_2 n}} 2^{-\binom{10\sqrt{\log_2 n}}{2}} > 1 - n^{-3}.$$

Thus, $|C| \leq 10\sqrt{\log_2 n}$.

Thus, S is a maximal independent set of G of size at most $\log_{\frac{1}{\frac{1}{2}+\alpha}} n + O\left((\ln \ln n)^2\right) + 10\sqrt{\log_2 n}$ with probability greater than $(1 - n^{-3})^3 > 1 - n^{-2}$. Note that

$$\log_{\frac{1}{\frac{1}{2}+\alpha}} n + O\left((\ln \ln n)^2\right) + 10\sqrt{\log_2 n} \leq (1+\delta)\log_2 n$$

when $\delta = 18/\sqrt{\log_2 n}$ and n is large. Hence the lemma is proved. □

We create a First-Fit-partition by iterating removing small maximal independent sets S_i from G using Lemma 8. Set $G_1 = G$, and for $i > 1$, $G_i = G - \cup_{j \le i} S_j$. Let $n_i = |V(G_i)|$. We use the following important observation that G_i is a random graph distributed as $G_{n_i,1/2}$ since the edges of G_i were never considered when the independent sets $S_1, S_2, \ldots, S_i$ were removed. By Lemma 8, there exists a maximal independent set S_i in G_i of size at most $(1+\delta)\log_2 n_i$ with probability greater than $1 - n_i^{-2}$. Let G_{i+1} be the subgraph of G_i induced by $V(G_i) \setminus S_i$. We iterate until $|V(G_t)| < n^{1/2}\log_2 n$. After that point, we greedily partition the remaining vertices into independent sets. Clearly, $\chi_{FF}(G) \ge t$. Note that

$$\begin{aligned}\mathbb{P}r\left[|S_i| < (1+\delta)\log_2 n_i \text{ for all } 1 \le i \le t\right] &\ge 1 - t\left(n^{1/2}\log_2 n\right)^{-2} \\ &\ge 1 - n\left(\frac{1}{n\left(\log_2 n\right)^2}\right) = 1 - \frac{1}{(\log_2 n)^2}.\end{aligned}$$

Hence,

$$\chi_{FF}(G) \ge t \ge \frac{n - n^{1/2}\log_2 n}{(1+\delta)\log_2 n} = \frac{1}{1+\delta}\left(\frac{n}{\log_2 n}\right) - \frac{1}{1+\delta}n^{1/2} > (1-2\delta)\frac{n}{\log_2 n}$$

for $\delta = 18/\sqrt{\log_2 n}$ and as $n \to \infty$. □

Note that we have not tried to optimize the constants appearing in Theorem 3.

3.1. **Concentration.** We next consider the concentration of the First-Fit chromatic number of random graphs.

Proposition 9. *Let G be a graph, and H a subgraph of G formed by deleting one edge from G. Then $|\chi_{FF}(G) - \chi_{FF}(H)| \le 1$.*

Proof. Let $k = \chi_{FF}(G)$, and let $\langle V_1, V_2, \ldots, V_k\rangle$ be a First-Fit partition of $V(G)$. Let e be the edge of G missing from H with endpoints u and v. Let V_i contain u and V_j contain v, where $i < j$ (note that $i \ne j$ since V_i is independent). We create a First-Fit partition for H by greedily coloring the vertices of $V(H) - V_j$ first (in the order given by $V_1, V_2, \ldots, V_{j-1}, V_{j+1}, \ldots, V_k$) and then by coloring the vertices of V_j. This creates a First-Fit partition with at least $k-1$ parts. Hence $\chi(H) \ge \chi(G) - 1$.

Similarly, let $\ell = \chi_{FF}(H)$, and let $\langle V_1, V_2, \ldots, V_\ell\rangle$ be a First-Fit partition of $V(H)$. Let e be the edge of G missing from H with endpoints u and v. Let V_i contain u and V_j contain v, where $i \le j$. Again, we create a First-Fit partition for G by greedily coloring the vertices of $V(G) - V_j$ first (in the order given by $\langle V_1, V_2, \ldots, V_{j-1}, V_{j+1}, \ldots, V_\ell\rangle$) and then by coloring the vertices of V_j. This creates a First-Fit partition with at least $\ell - 1$ parts. Hence $\chi(G) \ge \chi(H) - 1$. □

Proposition 10. *Let G be a graph, and H a subgraph of G formed by deleting one vertex from G. Then $\chi_{FF}(H) \le \chi_{FF}(G) \le \chi_{FF}(H) + 1$.*

Proof. Let v be the vertex deleted from G to form H. Let $\ell = \chi_{FF}(H)$, and let $V_1, V_2, \ldots, V_\ell$ be a First-Fit partition of $V(H)$. We create a First-Fit partition for G by greedily coloring the vertices of $V(H)$ first in the order given by $\langle V_1, V_2, \ldots, V_\ell \rangle$, and then by coloring v. This creates a First-Fit partition with at least ℓ parts. Hence $\chi(G) \geq \chi(H)$.

Let $k = \chi_{FF}(G)$, and let $V_1, V_2, \ldots, V_k$ be a First-Fit partition of $V(G)$. Let V_j contain v. We create a First-Fit partition for H by greedily coloring the vertices of $V(H) - V_j$ first (in the order given by $V_1, V_2, \ldots, V_{j-1}, V_{j+1}, \ldots, V_k$) and then by coloring the vertices of V_j. This creates a First-Fit partition with at least $k-1$ parts. Hence $\chi(H) \geq \chi(G) - 1$. □

Note that there are graphs whose first fit chromatic number drops when an edge is added. The smallest such example is P_4, which has $\chi_{FF}(P_4) = 3$, while $\chi_{FF}(C_4) = 2$.

Since the First-Fit chromatic number satisfies the edge Lipschitz condition, we obtain a tight concentration via the vertex exposure martingale and Azuma's inequality (see [1, pp 95-96]):

Theorem 11. *Let $G = G_{n,p}$ for any probability p. Then*

$$\mathbb{P}r\left[\left|\chi_{FF}(G) - \mathbb{E}[\chi_{FF}(G)]\right| > \lambda\sqrt{n-1}\right] < 2e^{-\lambda^2/2}.$$

This concentration result is analogous to the result of Shamir and Spencer [1] about the tight concentration of the chromatic number of the random graph.

4. First-Fit Chromatic Number of Product of Graphs

In this section we will discuss the First-Fit chromatic number of $G\square P_2$, $G\square P_3$ and $G\square P_4$.

Theorem 12. *Let $\chi_{FF}(G)$ be even. Then*

$$\chi_{FF}(G\square P_3) \geq \chi_{FF}(G) + 2.$$

Proof. For simplicity let $V(P_3) := \{1, 2, 3\}$. Suppose $\chi_{FF}(G) = 2k$. Then there exists a First-Fit partition of $V(G)$ into $2k$ non-empty sets $\langle V_1, V_2, \ldots, V_{2k} \rangle$. We construct a partition $\langle U_1, \ldots, U_{2k+2} \rangle$ of $V(G\square P_3)$ as follows:

(1) for $1 \leq i \leq k-1$, let $U_{2i-1} = \{(u,1), (u,3), (v,2) \mid u \in V_{2i-1}, v \in V_{2i}\}$, and $U_{2i} = \{(v,1), (v,3), (u,2) \mid u \in V_{2i-1}, v \in V_{2i}\}$,
(2) for $k \leq i \leq k+1$, choose two adjacent vertices $u \in V_{2k-1}$ and $v \in V_{2k}$. Let $U_{2k-1} = \{(u,3), (v,1)\}$, $U_{2k} = \{(u,1), (v,3)\}$, $U_{2k+1} = \{(u,2)\}$, $U_{2k+2} = \{(v,2)\}$.
(3) for the remaining vertices, fit them into the previous parts greedily.

Since the V_i $(1 \leq i \leq 2k)$ are all independent, it is easy to see that all U_i $(1 \leq i \leq 2k+2)$ are also independent.

For each $1 \leq j < i \leq k-1$ and each $u \in V_{2i-1}$, $v \in V_{2i}$, there exist vertices $x_1, y_1 \in V_{2j-1}$ and vertices $x_2, y_2 \in V_{2j}$ such that x_1, x_2 are neighbors of u and y_1, y_2 are neighbors of v.

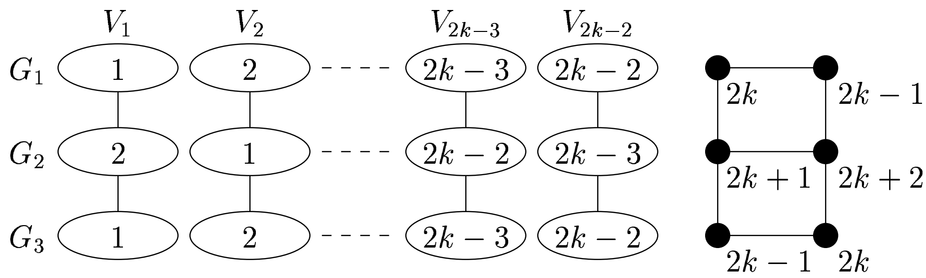

FIGURE 2. Partition of $V(G\Box P_3)$

Thus by the construction, (u,a) is adjacent to $(x_1,a)\in U_{2j-1}$ and $(x_2,a)\in U_{2j}$ for $a=1,3$ and $(u,2)$ is adjacent to $(x_2,2)\in U_{2j-1}$ and $(x_1,2)\in U_{2j}$.

Similarly, (v,a) is adjacent to $(y_1,a)\in U_{2j-1}$ and $(y_2,a)\in U_{2j}$ for $a=1,3$ and $(v,2)$ is adjacent to $(y_2,2)\in U_{2j-1}$ and $(y_1,2)\in U_{2j}$.

We can also find a vertex $u_1\in V_{2i-1}$ that is a neighbor of v. Then for any vertex in U_{2j}, $(v,1),(v,3)$ is adjacent to $(u_1,1),(u_1,3)\in U_{2j-1}$ and $(u,2)$ is adjacent to $(u,1)\in U_{2j-1}$. Hence any vertex in $U_i, 1\le i\le 2k-2$ has a neighbor in all the lower indexed subsets.

Similarly, we can check that any vertex in $U_{2k-1},U_{2k},U_{2k+1},U_{2k+2}$ has a neighbor in the lower indexed subsets.

Thus $\langle U_1,\dots,U_{2k+2}\rangle$ is a First-Fit partition of $G\Box P_3$, so $\chi_{FF}(G\Box P_3)\ge\chi_{FF}(G)+2$ if $\chi_{FF}(G)$ is even. □

We next consider the First-Fit chromatic number of $G\Box P_4$.

Theorem 13. *For every graph G,*

$$\chi_{FF}(G\Box P_4)\ge\chi_{FF}(G)+2.$$

Proof. Suppose $\chi_{FF}(G)=t$.

If t is even, since $G\Box P_3$ is an induced subgraph of $G\Box P_4$, hence $\chi_{FF}(G\Box P_4)\ge\chi_{FF}(G\Box P_3)\ge\chi_{FF}(G)+2$.

If $t=2k-1$ is odd, then we can make a partition similar to the one in the proof of Theorem 12 as follows:

(1) For $1\le i\le k-1$, let $U_{2i-1}=\{(u,1),(u,3),(v,2),(v,4)|\ u\in V_{2i-1},v\in V_{2i}\}$, and $U_{2i}=\{(v,1),(v,3),(u,2),(u,4)|\ u\in V_{2i-1},v\in V_{2i}\}$, where $\{1,2,3,4\}$ is the vertex set of P_4.
(2) Choose a vertex $u\in V_{2k-1}$. Let $U_{2k-1}=\{(u,1),(u,4)\}$, $U_{2k}=\{(u,3)\}$, $U_{2k+1}=\{(u,2)\}$.
(3) Color the remaining vertices greedily.

It is easy to check that the above partition is a First-Fit partition following an argument similar to the one in the proof of Theorem 12. □

Theorem 14. [4] $\chi_{FF}(C_{m_1}\Box C_{m_2}\Box\ldots\Box C_{m_k}) = 2k+1$, *where* $k \geq 2$, $m_i \geq 4$ *and* m_i *is even.*

Corollary 15. *Let* $C_{m_1}, C_{m_2}, \ldots, C_{m_k}$ *be cycles of size* $m_1, m_2, \ldots, m_k$ *respectively, where* $k \geq 2$ *and* $m_i \geq 4$*. Then*

$$\chi_{FF}(C_{m_1}\Box C_{m_2}\Box\ldots\Box C_{m_k}) = 2k+1.$$

Proof. The upper bound follows from the obvious degree bound, $\chi_{FF}(G) \leq \Delta(G)+1$. For the lower bound, we first reorder m_i's to place the even m_i's first, followed by the odd ones.

Case 1: If no m_i is odd, then the corollary follows by Theorem 14.

Case 2: If we have $k_1 \geq 2$ even m_i's, then the product graph of k_1 even cycles has FF-chromatic number $2k_1+1$ by Theorem 14. Then for the following odd cycles, we can repeatedly apply Theorem 13.

If we have at most one even cycle which is not C_4, then by the fact that $\chi_{FF}(C_n) = 3$ for $n \geq 5$ and repeatedly applying Theorem 13 proves the corollary. If the sole even cycle is C_4, then we use the fact that $\chi_{FF}(C_4\Box P_4) = 5$ and again repeatedly apply Theorem 13. □

In the following example we show that $\chi_{FF}(G\Box P_2)$ can be as large as $2\chi_{FF}(G)$. Since $G\Box P_2$ is an induced subgraph of $G\Box P_3$ and $G\Box P_4$, $\chi_{FF}(G\Box P_3)$ and $\chi_{FF}(G\Box P_4)$ can be as large as $2\chi_{FF}(G)$ as well.

Proposition 16. *Let* G *be a graph formed by removing a prefect matching* M *from* K_{2m}*. Then* $\chi_{FF}(G\Box P_2) = 2\chi_{FF}(G)$*.*

Proof. Suppose $V(K_{2m}) = \{u_1, \ldots, u_{2m}\}$ and the deleted matching is $M = \{u_1u_2, \ldots, u_{2m-1}u_{2m}\}$. Consider a First-Fit partition $\langle V_1, \ldots, V_t\rangle$ of $V(G)$, where $\chi_{FF}(G) = t$. Without loss of generality, suppose $u_1 \in V_1$. Then except u_2, all other vertices are adjacent to u_1, so they cannot be in V_1. Then u_2 must be in V_1, otherwise it does not have any neighbor in V_1. So $V_1 = \{u_1, u_2\}$. Inductively, we can see that $V_i = \{u_{2i-1}, u_{2i}\}$. So $\chi_{FF}(G) = t = m$. For $G\Box P_2$, we can make a partition as follows:

> for $1 \leq i \leq t$, $U_{2i-1} = \{(u_{2i-1}, 1), (u_{2i}, 2)\}, U_{2i} = \{(u_{2i-1}, 2), (u_{2i}, 1)\}$, where $\{1, 2\}$ is the vertex set of P_2.

So $\chi_{FF}(G\Box P_2) \geq 2\chi_{FF}(G)$. For the upper bound, $\chi_{FF}(G\Box P_2) \leq \Delta(G\Box P_2)+1 = (2m-1)+1 = 2m$. Hence $\chi_{FF}(G\Box P_2) = 2\chi_{FF}(G)$ in this case. □

We believe that $\chi_{FF}(G\Box P_2) \leq 2\chi_{FF}(G)$ for every graph G.

Acknowledgement:

The authors would like to thank Zoltán Füredi for his encouragement and helpful discussions, and Weiting Cao for helpful discussions during the early stages of this project.

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