

Problem 1

(a) We always start with $x^0 = (\frac{1}{n}, \dots, \frac{1}{n})$, so

$$f(z|x^0) = \left(\frac{nz_1}{\sum_i nz_i}, \dots, \frac{nz_n}{\sum_i nz_i} \right) = \left(\frac{nz_1}{n \cdot 1}, \dots, \frac{nz_n}{n \cdot 1} \right) = z.$$

Hence also $f^{-1}(z|x^0) = z$, so in particular $x^1 = f^{-1}(y^1|x^0) = y^1$.

(b) In general $x^2 \neq y^2$, as we will see in the next example.

(c) The problem we are solving has the following input:

$$A = \begin{pmatrix} 1 & 1 & -2 \end{pmatrix} \quad c = (0, 2, 0) \quad x^0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right)$$

In the first iteration we compute

$$x^1 = y^1 = \left(\frac{1}{3} + \frac{1}{8\sqrt{3}}, \frac{1}{3} - \frac{1}{8\sqrt{3}}, \frac{1}{3} \right)$$

The second iteration gives

$$\begin{aligned} y^2 &= \left(\frac{1}{3} - \frac{1}{24\sqrt{65}} + \frac{1}{\sqrt{195}}, \frac{1}{3} - \frac{1}{24\sqrt{65}} - \frac{1}{\sqrt{195}}, \frac{1}{3} + \frac{1}{12\sqrt{65}} \right) \\ x^2 &= \left(\frac{(8 + \sqrt{3})(520 - \sqrt{65} + 8\sqrt{195})}{48(260 + \sqrt{65})}, \frac{(-8 + \sqrt{3})(-520 + \sqrt{65} + 8\sqrt{195})}{48(260 + \sqrt{65})}, \frac{1}{3} \right) \neq y^2 \end{aligned}$$

(d) As we see in the above example, rationality does not have to be maintained.

Problem 2

(a) Assign weight $\binom{k}{2}^{-1}$ to all red edges and weight $\frac{1}{\lceil k/2 \rceil \lfloor k/2 \rfloor}$ to all blue edges. Since every red k -subgraph can have at most $\binom{k}{2}$ edges, and every blue graph is bipartite (so it has no more than $\lceil k/2 \rceil \lfloor k/2 \rfloor$ edges), this is a feasible solution for $r(c_1; n, k)$. There are $\binom{\lceil n/2 \rceil}{2} + \binom{\lfloor n/2 \rfloor}{2}$ red edges and $\lceil n/2 \rceil \lfloor n/2 \rfloor$ blue edges, so the total weight is

$$\frac{\binom{\lceil n/2 \rceil}{2} + \binom{\lfloor n/2 \rfloor}{2}}{\binom{k}{2}} + \frac{\lceil n/2 \rceil \lfloor n/2 \rfloor}{\lceil k/2 \rceil \lfloor k/2 \rfloor} \sim \frac{n^2}{4\binom{k}{2}} + \frac{n^2}{4\lceil k/2 \rceil \lfloor k/2 \rfloor}.$$

(b) Assign weight 0 to all edges inside V_i 's and weight $\frac{1}{\lceil k/2 \rceil \lfloor k/2 \rfloor}$ to all other edges. A mono- k -subgraph intersecting two, three, four or five of V_i 's can have at most $\frac{k^2}{4}, 2 \cdot \frac{k^2}{9}, 3 \cdot \frac{k^2}{16}$ or $5 \cdot \frac{k^2}{25}$ edges respectively, so this weight assignment is easily seen to be a feasible solution for $r(c_2; n, k)$. There are roughly $n^2/5$ red edges no internal to any V_i , and roughly $n^2/5$ blue edges of the same type, so the total weight is

$$\frac{2n^2}{5\lceil k/2 \rceil \lfloor k/2 \rfloor}$$

(c) As k tends to ∞ , the solution to part (a) is asymptotically $\frac{3}{2} \frac{n^2}{k^2}$, while the solution to part (b) is asymptotically $\frac{8}{5} \frac{n^2}{k^2}$, so the K_5 blow-up construction is better in general. When k is small, some additional computations are necessary.

We consider two cases depending on the parity of k . Assume without loss of generality that n is divisible by 10, so the above formulas are exact. First let k be even. After comparing the total weights obtained above we conclude that the value in (a) is better for $k = 2$ and $k = 4$, and the value in (b) is better for all even $k \geq 6$. When k is odd, the value obtained in (a) is better only for $k < 5$.

We conclude that the value obtained in (a) is better if $k < 5$ and the value obtained in (b) is better for all other values of k , i.e. $k \geq 5$.

(d) The constraints in the LP treat red and blue edges separately, so a weight assignment is optimal if and only if it maximizes sum of weights of blue edges and (independently) sum of weights of red edges. Let w be a feasible solution to $r(c_1; n, k)$. For a graph H let $w(H)$ denote the sum of the weights of all edges of H and let $l = \lfloor n/2 \rfloor$. Every edge in V_1 is contained in exactly $l - 2$ triangles, hence summing weights of all triangles in V_1 (having in mind that for each triangle T we have a constraint $w(T) \leq 1$) yields

$$\binom{l}{3} \geq \sum_T w(T) = (l - 2) \sum_{e \in G[V_1]} w(e).$$

Simplifying we get

$$\sum_{e \in G[V_1]} w(e) \leq \frac{1}{3} \binom{l}{2},$$

which is precisely the value of the optimal solution from (a), restricted to the edges inside V_1 . Performing analogous computation for V_2 , we conclude that the solution from (a) maximizes the sum of the weights of the red edges over all feasible solutions.

Now, let $[V_1, V_2]$ is the set of blue edges. Each induced blue-3-subgraph has one vertex in V_1 and two in V_2 or vice versa. Arguing similarly as above, for n even we get

$$2l \binom{l}{2} \geq \sum_T w(T) = 2(l - 1) \sum_{e \in [V_1, V_2]} w(e),$$

and when n is odd (and V_1 and V_2 have sizes l and $l + 1$), we obtain

$$l \binom{l + 1}{2} + (l + 1) \binom{l}{2} \geq \sum_T w(T) = (2l - 1) \sum_{e \in [V_1, V_2]} w(e).$$

Simplifying we get

$$\sum_{e \in [V_1, V_2]} w(e) \leq \frac{l^2}{2} \quad \sum_{e \in [V_1, V_2]} w(e) \leq \frac{l(l+1)}{2}$$

for n even and odd respectively. In both cases, this is exactly the value we got in part (a) for the blue subgraph. We conclude that the value we obtained in (a) is maximum over all feasible solutions.

(e) The above argument naturally generalizes to other values of k – we add weights of all red K_k 's and all blue “balanced” (i.e. $\lfloor k/2 \rfloor$ and $\lceil k/2 \rceil$ vertices in V_1 and V_2) blue graphs, each edge is counted the same number of times.