GENERALIZED R-COHESIVENESS AND THE ARITHMETICAL HIERARCHY: A CORRECTION TO “GENERALIZED COHESIVENESS”

CARL G. JOCKUSCH, JR.* AND TAMARA J. LAKINS†

ABSTRACT. For $X \subseteq \omega$, let $[X]^n$ denote the class of all n -element subsets of X . An infinite set $A \subseteq \omega$ is called *n-r-cohesive* if for each computable function $f : [\omega]^n \rightarrow \{0, 1\}$ there is a finite set F such that f is constant on $[A - F]^n$. We show that for each $n \geq 2$ there is no Π_n^0 set $A \subseteq \omega$ which is *n-r-cohesive*. For $n = 2$ this refutes a result previously claimed by the authors, and for $n \geq 3$ it answers a question raised by the authors.

1. INTRODUCTION

A generalized notion of cohesiveness, which arises in connection with effective versions of Ramsey’s theorem, was studied by Hummel and Jockusch in [1]. For any set X , let $[X]^n$ denote the class of all n -element subsets of X . A *k-coloring* f of $[X]^n$ is a function $f : [X]^n \rightarrow \{0, 1, \dots, k - 1\}$. A set $A \subseteq X$ is *homogeneous* for a coloring f of $[X]^n$ if $f \upharpoonright [A]^n$ is constant, i.e., if all n -element subsets of A are assigned the same color by f ; n is called the *exponent* of the coloring. An infinite version of Ramsey’s theorem states that for any infinite set X and any k -coloring f of $[X]^n$, there exists an infinite set $A \subseteq X$ which is homogeneous for f . A 2-coloring f of $[\omega]^n$ is called *computably enumerable* (or *c.e.*) if either $f^{-1}(0)$ or $f^{-1}(1)$ is c.e. when finite sets are identified with their canonical indices.

Definition 1.1.

- (1) A set A is *almost homogeneous* for a coloring f if there exists a finite set F such that $A - F$ is homogeneous for f .
- (2) An infinite set $A \subseteq \omega$ is *n-cohesive* (respectively, *n-r-cohesive*) if it is almost homogeneous for every computably enumerable (respectively, computable) 2-coloring of $[\omega]^n$.

It is easy to see that when $n = 1$, we obtain the usual definition of a cohesive or r-cohesive set. Thus, there exists a Π_1^0 1-cohesive set, i.e. a comaximal set (see [5], Theorem X.3.3).

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†Formerly Tamara J. Hummel.

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Jockusch [3] (Theorems 4.2 and 5.5) proved that for $n \geq 1$, every computable k -coloring of $[\omega]^n$ has an infinite Π_n^0 homogeneous set, and this result was shown to also hold for computably enumerable (c.e.) 2-colorings by Hummel and Jockusch [2] (Theorem 3.1). Both results are best possible for $n \geq 2$ by Theorems 3.1 and 5.1 of [3]. In addition, arithmetical n -cohesive sets exist; namely, Hummel and Jockusch [1] (Theorem 4.1) proved that for every $n \geq 1$, there exists a Δ_{n+1}^0 n -cohesive set. In that paper, we further claimed that there exists a Π_2^0 2-cohesive set [1] (Theorem 3.2) and asked for which $n \geq 1$ do Π_n^0 n -cohesive sets exist [1] (Question 3.7). In this paper, we refute our previous claim for the case $n = 2$ and answer the above question by proving that for no $n \geq 2$ does there exist a Π_n^0 n -r-cohesive set. Thus Theorem 4.1 of [1] is best possible with respect to the arithmetical hierarchy.

Our purported proof of the existence of a Π_2^0 2-cohesive set in [1] (Theorem 3.2) was based on a combination of two strategies. The first strategy was the strategy for constructing an infinite Π_2^0 homogeneous set for a single c.e. 2-coloring of $[\omega]^2$, modified to consider all c.e. 2-colorings of $[\omega]^2$.

The second strategy was similar to the strategy for constructing a maximal set, but used a modified notion of e -state, based on colorings of 1-element sets induced by the first strategy. However, it turns out that marker moves made to improve modified e -states in the second strategy can ruin the first strategy. In particular, the claim in the proof of Lemma 3.6 of [1] that there are infinitely many (i, k) -acceptable numbers is incorrect.

The authors discovered the error in this purported proof in the course of an attempt to prove that there exists a Π_n^0 n -cohesive set for each $n \geq 2$. The first step in this attempt was to develop a new construction of an infinite Π_3^0 homogeneous set for a single c.e. 2-coloring of $[\omega]^3$. This new construction seemed promising because it did not make use of the strong induction hypothesis involving retraceability used in the proof of Theorem 3.1 of [2], which is incompatible with the construction of a 3-cohesive set. However, in attempting to use this approach to construct a Π_3^0 3-cohesive set, they realized that there is a fundamental incompatibility between the strategy for making an infinite Π_3^0 homogeneous set and the strategy for making a maximal set. Furthermore, this incompatibility occurs in the exponent $n = 2$ case as well, and was so fundamental that it strongly suggested that the exponent $n = 2$ case should be false.

2. THE MAIN RESULT

Our main result will follow easily from the special case where $n = 2$, and this result is established in the following theorem.

Theorem 2.1. *If A is a Π_2^0 set, then A is not 2-r-cohesive.*

Proof. Let A be an infinite Π_2^0 set. We will define two computable functions $f, g : [\omega]^2 \rightarrow \{0, 1\}$ and prove that either A fails to be almost homogeneous for f or A fails to be almost homogeneous for g . Indices of these two functions will be obtained uniformly from a Π_2^0 index of A .

To define f and g we use a computable approximation to A . Let $\{A_s\}_{s \in \omega}$ be a uniformly computable sequence of sets such that, for all $n \in \omega$,

$$n \in A \iff (\exists^\infty s)[n \in A_s]$$

(where “ $\exists^\infty$ ” means “for infinitely many”). Such an approximation exists because A is Π_2^0 . Also define, for $n, s \in \omega$,

$$h(n, s) = |\{t < s : n \in A_t\}|.$$

For each $s \in \omega$, define a linear ordering $<_s$ of ω as follows:

$$a <_s b \iff (h(a, s) < h(b, s) \text{ or } (h(a, s) = h(b, s) \text{ and } a < b)).$$

The intuitive meaning of $a <_s b$ is roughly that at stage s , b seems more likely than a to belong to A . More precisely, we have the following:

$$\text{if } a \notin A \text{ and } b \in A, \text{ then } (\forall^\infty s)[a <_s b] \quad (*)$$

(where “ $\forall^\infty$ ” means “for all but finitely many”). The above is clear since if $a \notin A$ and $b \in A$, then $\lim_s h(a, s)$ exists and is finite, while $\lim_s h(b, s) = \infty$.

Now given a number a and a stage s , we attempt to approximate the least $b \in A$ such that $a <_s b$. More precisely, let $u(a, s)$ be the least k such that $a < k < s$ and $a <_s k$, if such a k exists, and otherwise let $u(a, s)$ be undefined. (Note: The words “least” and “greatest” refer to the standard ordering of ω (not $<_s$) unless otherwise specified.) Similarly, given b and s , we attempt to approximate the greatest $a \in A$ such that $a < b$, if such an a exists. More precisely, define $d(b, s)$ to be the greatest k such that $k < b$ and $b <_s k$, if such a k exists, and otherwise let $d(b, s)$ be undefined. Clearly the partial functions u and d (representing “up” and “down” respectively) are computable and have computable domains. There is no reason to think that u and d converge to their desired limits as s approaches infinity, but the following lemma gives a weaker result which suffices for our purposes.

Lemma 2.2. *Suppose that a and b are consecutive elements of A (in the standard ordering $<$ of ω) with $a < b$. Then there is a number s_0 such that for all $s \geq s_0$,*

- (1) *if $a <_s b$, then $u(a, s) = b$;*
- (2) *if $b <_s a$, then $d(b, s) = a$.*

Proof. Let s_0 be a stage such that $s_0 > b$ and for all $s \geq s_0$ and all $c < b$ with $c \notin A$, $c <_s a$ and $c <_s b$. Such a stage s_0 exists by (*). To show the first part, assume that $s \geq s_0$ and $a <_s b$. Then b is a number k such that $a < k < s$ and $a <_s k$. Furthermore, there is no number $c < b$ with these properties, since no such c is in A (as a, b are consecutive elements of A), and hence all such c satisfy $c <_s a$, by choice of s_0 . Thus b is the least number with these properties, so $u(a, s) = b$. The proof of the second part is entirely analogous. $\square$

The idea of defining the coloring $f : [\omega]^2 \rightarrow \{0, 1\}$ is to assign opposite colors to the pairs $\{b, s\}$ and $\{d(b, s), s\}$ whenever $b < s$ and $d(b, s)$ is defined.

This is easily achieved because if $d(b, s)$ is defined, then $d(b, s) < b < s$. Specifically, f is defined in stages as follows:

Stage s : There are substages $0, 1, \dots, s-1$ and $f(\{b, s\})$ is defined at substage b , for each b with $b < s$. At substage b , if $d(b, s)$ is defined, let $f(\{b, s\}) = 1 - f(\{d(b, s), s\})$, and otherwise let $f(\{b, s\}) = 0$. (Note that if $d(b, s)$ is defined, then $d(b, s) < b$ so $f(\{d(b, s), s\})$ was defined at the previous substage $d(b, s)$ of stage s .)

The idea of defining the coloring $g : [\omega]^2 \rightarrow \{0, 1\}$ is to assign opposite colors to the pairs $\{a, s\}$ and $\{u(a, s), s\}$ whenever $a < s$ and $u(a, s)$ is defined. This is easily achieved because if $u(a, s)$ is defined, then $a < u(a, s) < s$. The construction of g is analogous to that of f except that d is replaced by u and, for all $a < s$, $g(\{a, s\})$ is defined at substage $s - 1 - a$ of stage s , i.e. the values of $g(\{a, s\})$ for $a < s$ are defined at stage s in decreasing order of a . Thus, for $a < s$, at substage $s - 1 - a$ of stage s , if $u(a, s)$ is defined, let $g(\{a, s\}) = 1 - g(\{u(a, s), s\})$, and otherwise let $g(\{a, s\}) = 0$.

The following lemma suffices to complete the proof.

Lemma 2.3. *Either A is not almost homogeneous for f , or A is not almost homogeneous for g .*

Proof. Let $a <^* b$ mean that $a <_s b$ holds for infinitely many $s \in A$. We make no claim that $<^*$ is transitive, but for all a and b with $a \neq b$, either $a <^* b$ or $b <^* a$, because A is infinite. Let $a_0, a_1, \dots$ be a list of the elements of A in strictly increasing order. Clearly, either there are infinitely many n with $a_n <^* a_{n+1}$ or there are infinitely many n with $a_{n+1} <^* a_n$. Suppose there are infinitely many n with $a_{n+1} <^* a_n$. We claim that A is not almost homogeneous for f . To see this, it suffices to show that if $a_{n+1} <^* a_n$, then there are infinitely many $s \in A$ such that $\{a_n, a_{n+1}, s\}$ is not homogeneous for f . Suppose that $a_{n+1} <^* a_n$, so there are infinitely many $s \in A$ such that $a_{n+1} <_s a_n$. Then, by Lemma 2.2, $d(a_{n+1}, s) = a_n$ for all sufficiently large such s . Hence, by the construction of f , there are infinitely many $s \in A$ such that $f(\{a_n, s\}) \neq f(\{a_{n+1}, s\})$, and so $\{a_n, a_{n+1}, s\}$ is not homogeneous for f . By an analogous argument, one shows that if there are infinitely many n with $a_n <^* a_{n+1}$, then A is not almost homogeneous for g . $\square$

$\square$

We don't know whether it is possible to make the above proof more uniform by showing that there is an effective procedure for obtaining from a Π_2^0 index of an infinite set A a characteristic index of a *single* computable function f for which A fails to be almost homogeneous. If this could be done, it would then follow from the recursion theorem that there is no computable function h such that, for all e , if φ_e is a total function from $[\omega]^2$ to $\{0, 1\}$, then $h(e)$ is a Π_2^0 index of an infinite Π_2^0 set A which is almost homogeneous for φ_e . The result we have just proved amounts to showing that there is no such h which is constant.

The generalization to arbitrary n now comes cheaply.

Corollary 2.4. *If $n \geq 2$ there does not exist a Π_n^0 set A which is n -r-cohesive.*

Proof. The following lemma is implicit in the proof of Theorem 4.5 of [1].

Lemma 2.5. *Let A be any set and let $n \geq 2$. If a set B is n -r-cohesive relative to A , then B is $(n-1)$ -r-cohesive relative to A' .*

The proof of this lemma is a straightforward application of the Limit Lemma, which we omit here. It follows by induction on $n \geq 2$ that if a set B is n -r-cohesive relative to A , then A is 2-r-cohesive relative to $A^{(n-2)}$. Applying this with $A = \emptyset$, it follows that if A is n -r-cohesive, then A is 2-r-cohesive relative to $0^{(n-2)}$. But if A is Π_n^0 , then A is $\Pi_2^{0,0^{(n-2)}}$. However, it is easily seen that Theorem 2.1 holds relative to $0^{(n-2)}$, so there does not exist a Π_n^0 n -r-cohesive set for any $n \geq 2$. $\square$

As already mentioned, Theorem 2.1 shows that when $n \geq 2$, the best possible arithmetical complexity of an n -cohesive or n -r-cohesive set is Δ_{n+1}^0 . It thus follows that Theorem 4.15 of [1], which states that for all $n \geq 2$, there exists a Δ_{n+1}^0 n -r-cohesive set A such that $0^{(n-1)} \not\leq_T A$, is best possible. It remains open to determine for which $n \geq 2$ there is an n -cohesive set $A <_T 0^{(n)}$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 W. GREEN ST., URBANA, IL 61801

E-mail address: jockusch@math.uiuc.edu

DEPARTMENT OF MATHEMATICS, ALLEGHENY COLLEGE, 520 N. MAIN ST., MEADVILLE, PA 16335

E-mail address: tlakins@allegheny.edu