Math 447 **Hour Exam # 1** **February 11, 2008**

NAME KEY

Each problem is worth 25 points. The real numbers are denoted by **R**.

1. (i) (8 points) State the Bolzano-Weierstrass Theorem.

Every bounded sequence has a convergent subsequence.

(ii) (8 points) State the Monotone Subsequence Theorem.

Every sequence has a monotone subsequence.

(iii) (9 points) Define lim inf a_n where a_n is a sequence in **R**.

For $N = 1, 2, 3, \ldots$, let $A_N = \{a_n : n \geq N\}$.

Let $v_N = \inf A_N$. Then

$$\liminf a_n = \lim_{N \to \infty} v_N .$$

2. (i) (10 points) Define: a_n is a Cauchy sequence in **R**.

a_n is a Cauchy sequence if for every $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that if $n > N$ and $m > N$, then $|a_n - a_m| < \varepsilon$.

(ii) (15 points) Prove from the definition that if a_n and b_n are both Cauchy sequences in **R**, so is $a_n + b_n$.

Suppose $\varepsilon > 0$. $\exists\, N_1 = N_1(\varepsilon)$ such that if $n > N_1$ and $m > N_1$, then $|a_n - a_m| < \varepsilon/2$.

Also $\exists\, N_2 = N_2(\varepsilon)$ such that if $n > N_2$ and $m > N_2$, then $|b_n - b_m| < \varepsilon/2$.

Let $N = \max(N_1, N_2)$. Suppose $n > N$ and $m > N$.

Then $|a_n + b_n - (a_m + b_m)| = |a_n - a_m + b_n - b_m|$

$\leq |a_n - a_m| + |b_n - b_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

This shows $a_n + b_n$ is Cauchy.

3. (5 points each) Suppose a_n is a sequence in **R** such that $-3 < a_n < -1$ if n is odd, and $1 < a_n < 3$ if n is even. The given information tells us that $-3 \le \liminf a_n \le -1$ and $1 \le \limsup a_n \le 3$, and no more than that.

(i) Circle each of the following statements that MUST be true. You *need not* justify your answers.

(a) $\liminf a_n < \limsup a_n$.

(b) $\liminf a_n > -3$

(ii) Circle each of the following statements that MIGHT be true. You *need not* justify your answers.

(c) $\limsup a_n - \liminf a_n = 2$

(d) $\limsup a_n < -1$

(e) a_n is a Cauchy sequence

4. Suppose A and B are bounded subsets of **R** with the property that $a < b$ for every a in A and every b in B. Prove carefully that it must be true that

$$\sup A \le \inf B.$$

Proof: Fix an arbitrary $b \in B$. For every $a \in A$, we have $a < b$. Thus b is an upper bound for A. Hence $\sup A \le b$.

Since $b \in B$ is arbitrary, we conclude that $\sup A$ is a lower bound for B. Hence

$$\sup A \le \inf B$$