Math 447 **Hour Exam # 2** **March 26, 2008**

NAME KEY

Do any three of the four problems. Clearly indicate which problem is not to be graded.

You may use any results from class.

1. Determine whether each series converges or diverges. Justify your answer.

(i) $$\sum_{n=1}^{\infty}\left(\frac{n^n}{n!}\right)^{1/2}$$

Certainly $n^n \geq n!$. Thus $\left(\frac{n^n}{n!}\right)^{1/2} \geq 1$ for $n \geq 1$.

Since $\left(\frac{n^n}{n!}\right)^{1/2} \not\to 0$, the series diverges.

(ii) $$\sum_{n=2}^{\infty}\frac{1}{(n+(-1)^n)^2}$$

We have $n+(-1)^n \geqslant n-1 \geqslant \frac{n}{2}$ for $n \geqslant 2$.

Thus $\frac{1}{(n+(-1)^n)^2} \leq \left(\frac{2}{n}\right)^2 = \frac{4}{n^2}$

Since $\sum_{n=2}^{\infty}\frac{4}{n^2}$ converges $(p=2)$, our series converges by the Comparison Test.

2. Let (X,d) be a metric space. Suppose S is a subset of X.

(i) (10 points) Define what it means for S to be compact.

S is compact if every open cover of S has a finite subcover.

(ii) (23 points) Suppose S is a compact subset of X, and that F_n is a sequence of closed nonempty subsets of S such that $F_{n+1} \subset F_n$ for n = 1, 2, 3 ... Show that

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

(In class we showed this is true if X = $\mathbf{R}^k$ using the Heine-Borel Theorem and the Bolzano-Weierstrass Theorem. Here we have only the definition of compactness with which to work.)

Suppose $\bigcap_{n=1}^{\infty} F_n = \emptyset$ and seek a contradiction.

Let $U_n = X - F_n$. Note that each U_n is open.

Since $\bigcap_{n=1}^{\infty} F_n = \emptyset$, for each $x \in X$ there exists n such that $x \notin F_n$, i.e., $x \in U_n$. So $\bigcup_{n=1}^{\infty} U_n = X$.

Also $U_{n+1} \supset U_n$, $n = 1, 2, \dots$ Evidently the U_n's form an open cover of S and thus a finite number of the U_n's cover S. Since $U_n \subset U_{n+1}$ all n, we see in fact there exists N such that $S \subset U_N$. Then

$F_N \subset S \subset U_N = X - F_N$, a contradiction since $F_N \neq \emptyset$.

3. Suppose (X,d) is a metric space and S is a subset of X.

(i) (10 points) Define what it means for S to be path-connected.

S is path-connected if for each $p \in S$ and $q \in S$ there exists $[a,b] \subset \mathbb{R}$ and a continuous $\gamma : [a,b] \to S$ such that $\gamma(a) = p$ and $\gamma(b) = q$.

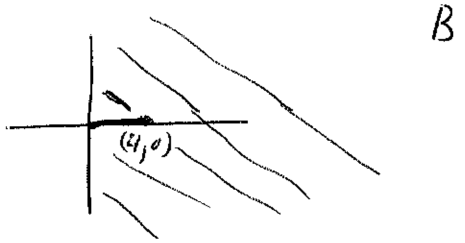
(ii) Consider the subset $B = B_1 \cup B_2 \cup B_3$ of $\mathbf{R}^2$ where

$$B_1 = \{(x_1, x_2) : x_1 > 0 \text{ and } x_2 > 0\}$$

$$B_2 = \{(x_1, x_2) : x_1 > 0 \text{ and } x_2 < 0\}$$

$$B_3 = \{(x_1, 0) : x_1 > 4\}.$$

(a) (5 points) Sketch B.



(b) (18 points) Is B connected? Take a definite stand. Outline a justification of your answer. It is not necessary to include all details.

Suppose p and q are both in B. Evidently the line segments joining p to $(5,0)$ and q to $(5,0)$ both lie in B. Thus there is a path in B from p to $(5,0)$ and on to q. This shows that B is path-connected. A theorem from class now implies B is connected.

4. Suppose $f: \mathbf{R} \to \mathbf{R}$.

(i) (10 points) Define what it means for f to be uniformly continuous on $(0, +\infty)$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x_1 \in (0, +\infty)$, $x_2 \in (0, +\infty)$, and $|x_1 - x_2| < \delta$, then

$$|f(x_2) - f(x_1)| < \varepsilon$$

(ii) (23 points) Suppose that in fact the above function f is continuous on $\mathbf{R}$ and

$$\lim_{x \to +\infty} f(x) = 2.$$

Must f be uniformly continuous on $(0, +\infty)$? Justify your answer by appealing to appropriate theorems from class.

Yes. Suppose $\varepsilon > 0$. There exists $M > 0$ such that if $x > M$ then $|f(x) - 2| < \varepsilon/2$. Thus if $x_1 > M$ and $x_2 > M$ then $|f(x_1) - f(x_2)| \leq |f(x_1) - 2| + |2 - f(x_2)| < \varepsilon$.

Since $[0, M+1]$ is compact, f is uniformly continuous on $[0, M+1]$. Thus there exists $\delta_1 > 0$ such that if $0 \leq x_j \leq M+1$ for $j = 1, 2$, and $|x_2 - x_1| < \delta_1$, then $|f(x_1) - f(x_2)| < \varepsilon$.

Let $\delta = \min(1, \delta_1)$. Suppose $x_1 > 0$, $x_2 > 0$, and $|x_2 - x_1| < \delta$.

(a) If either $x_1 > M+1$ or $x_2 > M+1$, then $x_1 > M$ and $x_2 > M$ and hence $|f(x_1) - f(x_2)| < \varepsilon$.

(b) If $0 < x_1 \leq M+1$ and $0 < x_2 \leq M+1$, then $|f(x_1) - f(x_2)| < \varepsilon$ since $\delta \leq \delta_1$.

Thus in all cases $|f(x_1) - f(x_2)| < \varepsilon$, establishing that f is uniformly continuous on $(0, +\infty)$.