Math 447 **Hour Exam # 3** **April 18, 2008**

NAME KEY

The problems are of equal weight. You may use any result from class.

1. Suppose $f : [a, b] \to \mathbf{R}$ is bounded.

(i) (15 points) State the Cauchy Criterion for integrability of f.

f is integrable on $[a,b]$ if and only if for every $\varepsilon > 0$, there exists a partition P such that $U(f, P) - L(f, P) < \varepsilon$.

(ii) (18 points) Suppose for some particular point c in $[a, b]$ that f has the form

$$f(x) = \begin{cases} 0, & x \in [a,b] - \{c\} \\ 1, & x = c. \end{cases}$$

Show that f is integrable on $[a, b]$ and evaluate the integral.

Let $P: a = x_0 < x_1 < \dots < x_n = b$ be any partition of $[a,b]$. Clearly $m_i = 0$, $1 \le i \le n$, and hence $L(f, P) = 0$. Thus $\underline{\int_a^b} f(x)dx = \sup_P L(f, P) = 0$.

Suppose $\varepsilon > 0$. Let Q be a partition of $[a,b]$ such that c belongs to exactly one of the subintervals determined by Q and such that mesh $Q < \varepsilon$. Then $M_i = 0$ for all but at most one value of i and the remaining $M_k = 1$. Thus $U(f, Q) \le$ mesh $Q < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude $\overline{\int_a^b} f(x)dx \le 0$. Thus

$$\overline{\int_a^b} f(x)dx \le 0 = \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx.$$ Thus f is integrable with integral 0.

2. Suppose that $f: \mathbf{R} \to \mathbf{R}$ is differentiable and that

$$f(1) = 2, \quad f(3) = 3, \quad \text{and} \quad f(5) = 10.$$

(i) (15 points) Show that there exists c in $(1,5)$ such that $f'(c) = 2$. Give a careful statement of any theorem from class that you apply.

MVT: If f is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c \in (a,b) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

We apply the MVT with $a=1$ and $b=5$ to conclude

$\exists\, c \in (1,5) \ni f'(c) = \dfrac{f(5)-f(1)}{5-1} = \dfrac{10-2}{4} = 2$ ✓

(ii) (18 points) Show that there exists d in $(1,5)$ such that $f'(d) = 1$. Give a careful statement of any theorem from class that you apply.

Apply the MVT on $[1,3]$; we conclude $\exists\, p \in (1,3)$

$\ni f'(p) = \dfrac{f(3)-f(1)}{3-1} = \dfrac{3-2}{3-1} = \dfrac{1}{2}$.

Intermediate Value Theorem for Derivatives:

If f is differentiable on an open interval containing p and c and L is a number between $f'(p)$ and $f'(c)$, then $\exists\, d$ between p and c such that $f'(d) = L$.

Since $f'(p) < 1 < f'(c)$, we apply the above theorem to conclude $\exists\, d$ between p and c (and hence lying in $(1,5)$) such that $f'(d) = 1$.

3. Suppose that $E \subset \mathbf{R}$, and $f_n : E \to \mathbf{R}$.

(i) Define what it means for $\sum_{n=1}^{\infty} f_n(x)$ to converge uniformly on E.

Let $F_p = \sum_{n=1}^{p} f_n(x)$. $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on E if there exists $F: E \to R$ such that for every $\varepsilon > 0$, $\exists N \ni$ if $p > N$ then $|F_p(x) - F(x)| < \varepsilon$ for all $x \in E$.

For the remaining two parts of this problem, consider the specific situation of $f_n : (0, +\infty) \to \mathbf{R}$ given by

$$f_n(x) = \frac{nx}{1 + n^3 x^3}.$$

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(ii) Does $\sum_{n=1}^{\infty} f_n(x)$ converge uniformly on $(0,1)$?

Justify your answer. No. Note that $f_n\left(\frac{1}{n}\right) = \frac{1}{2}$ and $\frac{1}{n} \in (0,1)$ for $n \geq 2$. Thus

$$\sup_{x \in (0,1)} |F_{p+1}(x) - F_p(x)| = \sup_{0<x<1} |f_{p+1}(x)| \geq 1/2 \quad \text{for } p \geq 1.$$

Thus $\{F_p(x)\}$ is not uniformly Cauchy on $(0,1)$ and hence does not converge uniformly on $(0,1)$

(iii) Does $\sum_{n=1}^{\infty} f_n(x)$ converge uniformly on $(1,+\infty)$?

Justify your answer. Yes. Note for $x>1$ that

$$0 < f_n(x) = \frac{nx}{1+n^3x^3} < \frac{nx}{n^3x^3} = \frac{1}{n^2x^2} < \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, we conclude from the Weierstrass M-test that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $(1,+\infty)$.