Math 447 **Final Examination** **May 3, 2008**

NAME KEY

Each problem is worth 25 points. You may use any results from class.

1. (i) Prove directly from the definition that a Cauchy sequence in **R** is bounded.

Since x_n is Cauchy, $\exists$ positive integer N such that if $n \geq N$ and $m \geq N$, then $|x_n - x_m| < 1$. Setting $m = N$, we see for all $n \geq N$ that $|x_n - x_N| < 1$, implying $|x_n| \leq |x_n - x_N| + |x_N| < 1 + |x_N|$. Thus for all n we have $x_n \leq \max\{|x_1|, |x_2|, \dots |x_{N-1}|, 1+|x_N|\}$. This says the sequence x_n is bounded.

(ii) Prove directly from the definition that $\lim_{x \to 3} 1/(2x + 1) = 1/7$.

Suppose $\varepsilon > 0$. Let $\delta = \min(1, \frac{35}{2}\varepsilon)$. Suppose $0 < |x-3| < \delta$. Then $|x-3| < 1$, implying $x > 2$ and $2x+1 > 5$. For these values of x we have

$$\frac{1}{2x+1} - \frac{1}{7} = \frac{7-2x-1}{7(2x+1)} = \frac{2(3-x)}{7(2x+1)}$$

Hence $$\left|\frac{1}{2x+1} - \frac{1}{7}\right| = \frac{2|x-3|}{7|2x+1|} < \frac{2|x-3|}{7 \cdot 5} < \varepsilon$$

Since $|x-3| < \frac{35}{2}\varepsilon$.

Thus $\lim_{x \to 3} \frac{1}{2x+1} = \frac{1}{7}$.

2. (i) Define: (a) a connected subset of the metric space (X , d) .

$E \subset X$ is connected if there do not exist sets A and B in X such that $A \cap B = \phi$, $A \cup B = E$, $A \neq \phi$, $B \neq \phi$, where A and B are of the form $A = E \cap U$ and $B = E \cap V$ for open subsets U and V of X,

(b) a continuous function from the metric space (X , d) into the metric space (Y , d*) .

$f: (X, d) \to (Y, d^*)$ is continuous if it is continuous at each point of X. f is continuous at x_0 in X if for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in B(x_0, \delta)$, then $f(x) \in B(f(x_0), \varepsilon)$.

(ii) State: (c) the Heine-Borel Theorem .

A subset E of R^k is compact if and only if it is closed and bounded.

(d) the Bolzano-Weierstrass Theorem (in **R**) .

Every bounded sequence in R has a convergent subsequence.

3. (i) (9 points) State either form of the Fundamental Theorem of Calculus.

FTC II: If $f: [a,b] \to R$ is integrable, then $F: [a,b] \to R$ given by $F(x) = \int_a^x f(t)\,dt$ is continuous and at each $c \in (a,b)$ that is a point of continuity of f, we have $F'(c) = f(c)$.

(ii) Suppose $f: [0,1] \to \mathbf{R}$ is continuous and let $F(x) = \int_0^x f(t)\,dt$, $0 \le x \le 1$.

Suppose further that $F(x)$ is rational for each x in $[0,1]$.

(a) Show that F is constant on $[0,1]$. Suppose $x_1 < x_2$ in $[0,1]$ and $F(x_1) \neq F(x_2)$. Seek ⊘ Let y_0 be an irrational number between $F(x_1)$ and $F(x_2)$. We know F is continuous by FTC II. Applying the Intermediate Value Theorem to F on $[x_1, x_2]$, we conclude there exists $c \in (x_1, x_2)$ such that $F(c) = y_0$, an irrational number. Contradiction!

(b) Draw the strongest conclusion you can about f(t) and justify your answer.

Since F is constant, $F'(x) \equiv 0$ on $(0,1)$. Since f is continuous we conclude for $0 < x < 1$ that $f(x) = F'(x) = 0$. By continuity at 0 and 1, $f(x) = 0$, $0 \le x \le 1$.

4. (i) Suppose $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R = 3$. Find the radius of convergence of $\sum_{n=0}^{\infty} b_n x^n$ where $b_n = n^5 2^n a_n$ and justify your answer.

Note $|b_n|^{\frac{1}{n}} = n^{\frac{5}{n}} \cdot 2 \cdot |a_n|^{\frac{1}{n}}$. We know $2 \cdot n^{\frac{5}{n}} \to 2 \cdot 1 = 2$.

From a result in class, $\overline{\lim}\, |b_n|^{\frac{1}{n}} = 2 \cdot \overline{\lim}\, |a_n|^{\frac{1}{n}} = 2 \cdot \frac{1}{3} = \frac{2}{3}$

Thus $\tilde{R}$ for $\sum_{n=0}^{\infty} b_n x^n$ is $\tilde{R} = \frac{1}{\overline{\lim}\, |b_n|^{\frac{1}{n}}} = \boxed{\frac{3}{2}}$

(ii) Show that $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges.

$\frac{1}{n2^n} \leq \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series. Thus $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges by the comparison test.

(iii) Find the exact value of the series in (ii). Hint: Consider $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

You may use familiar facts from calculus not discussed in this course.

geometric series: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ $\quad |x|<1$

integrate term-by-term:

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad |x|<1$$

(note these functions agree at $x=0$)

Set $x = \frac{1}{2}$: $\quad -\ln \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{n2^n}$

$''$

$\boxed{\ln 2}$

5. In this problem you may use the following facts about the sine function:

(a) $2t/\pi < \sin t, \quad 0 < t < \pi/2,$

(b) $|\sin t| \leq 1, \quad 0 < t.$

Let $f_n(x) = \dfrac{\sin nx}{1 + nx}, \quad x > 0.$

(i) Show $f_n \to 0$ uniformly on $[a, 10]$ for every a in $(0,10)$.

Suppose $\varepsilon > 0$. Let $N = \frac{1}{a\varepsilon}$. Then if $n > N$ we have for all $x \geq a$ that $|f_n(x)| \leq \dfrac{1}{1+na} < \dfrac{1}{na} < \varepsilon$.

Thus $f_n \to 0$ uniformly on $[a, +\infty)$ in fact.

(ii) Show f_n does not converge to 0 uniformly on $(0,10)$.

Note that $f_n\left(\frac{1}{n}\right) = \dfrac{\sin 1}{1+1} = \frac{1}{2} \sin 1 > \frac{1}{\pi}$ for all n.

Thus $f_n \not\to 0$ uniformly on $(0, 10)$.

(iii) Evaluate $\lim\limits_{n \to \infty} \displaystyle\int_0^{10} f_n(x)\, dx$ and justify your answer.

Note $|f_n(x)| \leq 1$ for all $x \geq 0$ and all n. Suppose $\varepsilon > 0$.

Since $f_n \to 0$ uniformly on $[\frac{\varepsilon}{2}, 10]$, we know that $\lim\limits_{n \to \infty} \displaystyle\int_{\varepsilon/2}^{10} f_n(x)\,dx = 0$. Then $\exists\, N \ni$ if $n > N$ that $\left|\displaystyle\int_{\varepsilon/2}^{10} f_n(x)\,dx\right| < \varepsilon/2$.

For such n, $\left|\displaystyle\int_0^{10} f_n(x)dx\right| = \left|\displaystyle\int_0^{\varepsilon/2} f_n(x)dx + \int_{\varepsilon/2}^{10} f_n(x)dx\right|$

$$\leq \left|\int_0^{\varepsilon/2} f_n(x)\,dx\right| + \left|\int_{\varepsilon/2}^{10} f_n(x)dx\right| \leq \frac{\varepsilon}{2} \cdot 1 + \frac{\varepsilon}{2} = \varepsilon$$

$|f_n(x)| \leq 1$ $\quad (n > N)$

$\boxed{\text{Thus } \lim\limits_{n \to \infty} \int_0^{10} f_n(x)\,dx = 0}$

6. Suppose $f: [0, +\infty) \to \mathbf{R}$ is continuous and that

(a) $f'(x)$ exists for $x > 0$,

(b) $f(0) = 0$,

(c) f' is monotone increasing on $(0, +\infty)$.

Let $g(x) = f(x)/x$. Show that g is monotone increasing on $(0, +\infty)$.

Note that $g'(x) = \dfrac{x f'(x) - f(x)}{x^2}$.

Apply the Mean Value Theorem to f on $[0, x]$ to conclude there exists c in $(0, x)$ such that

$$f(x) = f(x) - f(0) = f'(c)(x - 0) = f'(c)x$$

Since f' is increasing we conclude

$$f(x) = f'(c)x < f'(x)x$$

This tells us that $g'(x) > 0$ for $x > 0$ and hence g is increasing on $(0, +\infty)$

7. Let $A = \{(x_1, x_2) : 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 < 1\}$.

(i) Show that A is not a compact subset of $\mathbf{R}^2$ by exhibiting an open cover of A having no finite subcover.

Among many such examples is $\{U_n : n \in N\}$

where $U_n = \{(x_1, x_2) : x_2 < 1 - \frac{1}{n}\}$

(ii) Give an example of a continuous real-valued function on A that is not uniformly continuous. You *need not* justify your answer.

Perhaps the easiest example is

$f(x_1, x_2) = \frac{1}{1 - x_2}$

(iii) Does there exist a continuous $g : \bar{A} \to \mathbf{R}$ whose range is $[0, +\infty)$? Here A denotes the closure of A. Justify your answer.

No. $\bar{A}$ is compact by the Heine-Borel Theorem.

(It is clear that $\bar{A} = \{(x_1, x_2) : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 1\}$.

Hence $g(\bar{A})$ must be a compact subset of R.

But $[0, +\infty)$ is unbounded and hence not compact, again by the Heine-Borel Theorem.

8. Suppose $f:[a,b] \to (-5,5)$ is continuous on $[a,b]$ except for a single point c in (a,b).

(i) Show by example that there need not exist a continuous

$g:[a,b] \to [-5,5]$ such that $g(x) = f(x)$ for all x in $[a,b]$

except $x = c$.

Here is one example: $f(x) = \begin{cases} 2 & a \le x \le \frac{a+b}{2} \\ -2 & \frac{a+b}{2} < x \le b \end{cases}$

If $g(x) = f(x)$ for all $x \in [a,b]$ except $c = \frac{a+b}{2}$,

clearly $\lim_{x \to c} g(x) = \lim_{x \to c} f(x)$ does not exist.

Thus g cannot be continuous at c.

(ii) Show that f is integrable on $[a,b]$.

Suppose $\varepsilon > 0$. WLOG we suppose $\frac{\varepsilon}{60} < \min(c-a, b-c)$.

Then f is integrable on interval $I_1 = [a, c - \frac{\varepsilon}{60}]$

since it is continuous there. Thus $\exists$ partition

P_1 of I_1 $\ni$ $U(f, P_1) - L(f, P_1) < \varepsilon/3$. Likewise there

exists a partition P_2 of interval $I_2 = [c + \frac{\varepsilon}{60}, b]$

such that $U(f, P_2) - L(f, P_2) < \varepsilon/3$. Set $P = P_1 \cup P_2$.

Note that P is a partition of $[a,b]$. We have

$U(f,P) - L(f,P) \le U(f,P_1) - L(f,P_1) + U(f,P_2) - L(f,P_2) +$

$+ 5 \cdot \frac{2\varepsilon}{60} + 5 \cdot \frac{2\varepsilon}{60} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$.

By the Cauchy Criterion, f is integrable on $[a,b]$.