(23.4) (a) Note if n is even, $a_n^{\frac{1}{n}} = 6/5$

Note if n is odd, $a_n^{\frac{1}{n}} = 2/5$

Thus $\overline{\lim}\, a_n^{\frac{1}{n}} = 6/5$ and $\underline{\lim}\, a_n^{\frac{1}{n}} = 2/5$

Note if n is even, $\frac{a_{n+1}}{a_n} = \frac{(2/5)^{n+1}}{(6/5)^n} = \frac{2}{5}\left(\frac{1}{3}\right)^n \to 0$

Note if n is odd, $\frac{a_{n+1}}{a_n} = \frac{(6/5)^{n+1}}{\left(\frac{2}{5}\right)^n} = \frac{6}{5}\cdot 3^n \to +\infty$

Thus $\overline{\lim}\, \frac{a_{n+1}}{a_n} = +\infty$ and $\underline{\lim}\, \frac{a_{n+1}}{a_n} = 0$

(b) For even n, $a_n = (6/5)^n > 1$. Thus $a_n \not\to 0$

and so both $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} (-1)^n a_n$ diverge.

(c) $R = \frac{1}{\overline{\lim}\, |a_n|^{\frac{1}{n}}} = 5/6$

If $|x| = 5/6$, then for even n, $|a_n x^n| = \left(\frac{6}{5}\right)^n\left(\frac{5}{6}\right)^n = 1$.

Since $a_n x^n \not\to 0$, $\sum_{n=0}^{\infty} a_n x^n$ diverges if $x = \pm\frac{5}{6}$.

So $\sum_{n=0}^{\infty} a_n x^n$ converges if and only if

$$-\frac{5}{6} < x < \frac{5}{6}.$$

(24.17) Note we know from class that f is continuous on $[a,b]$. Suppose $\varepsilon > 0$. $\exists N \ni n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon/2$ for $a \leq x \leq b$. Since f is continuous at x, $\exists N_1 \ni n > N_1 \Rightarrow |f(x_n) - f(x)| < \varepsilon/2$. Suppose $n > \max(N, N_1)$. Then

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

where the first inequality holds since $n > N$ and the second holds since $n > N_1$.

(25.5) Since $f_n \to f$ uniformly on S, $\exists N_0 \ni |f_{N_0}(x) - f(x)| < 1$ for all $x \in S$. Since f_{N_0} is bounded on S, $\exists M > 0 \ni |f_{N_0}(x)| < M$ for all $x \in S$. Then if $x \in S$, we have

$$|f(x)| \leq |f(x) - f_{N_0}(x)| + |f_{N_0}(x)|$$

$$< 1 + M.$$

Thus f is bounded on S.