Math 447 Homework #12 Answers

(28.4) For $x \neq 0$, by the usual differentiation rules, we have $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$. Meanwhile,

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x\sin\frac{1}{x} = 0$$

since $\sin\frac{1}{x}$ is bounded. Thus f is differentiable everywhere. But f' is not continuous at 0 since $\lim_{x\to 0} f'(x)$ does not exist. For if $\lim_{x\to 0} f'(x) = L$, then

$$\lim_{x\to 0} \cos\frac{1}{x} = \lim_{x\to 0} 2x\sin\frac{1}{x} - f'(x) = 0 - L,$$

a contradiction since $\lim_{x\to 0}\cos\frac{1}{x}$ does not exist.

(28.15) We prove

$$(fg)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a)\, g^{(n-k)}(a) \qquad (*)$$

by mathematical induction.

I. If $n=1$, $*$ asserts that

$$(fg)'(a) = \binom{1}{0} f'(a)g'(a) + \binom{1}{1} f(a)g(a)$$

$$= f(a)g'(a) + f'(a)g(a)$$

This is just the product rule. ✓

II. Suppose $*$ holds for n. We show it then must hold for $n+1$.

We have $(fg)^{(n+1)}(a) = \left((fg)^{(n)}\right)'(a)$

$$= \sum_{k=0}^{n} \binom{n}{k} \left\{ f^{(k+1)}(a) g^{(n-k)}(a) + f^{(k)}(a) g^{(n+1-k)}(a) \right\}$$

$$= \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)}(a) g^{(n-k)}(a) + \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(a) g^{(n+1-k)}(a)$$

$j = k+1$ $\qquad\qquad$ $j = k$

$$= \sum_{j=1}^{n+1} \binom{n}{j-1} f^{(j)}(a) g^{(n+1-j)}(a) + \sum_{j=0}^{n} \binom{n}{j} f^{(j)}(a) g^{(n+1-j)}(a)$$

$$= \binom{n}{n} f^{(n+1)}(a) g^{(0)}(a) + \binom{n}{0} f^{(0)}(a) g^{(n+1)}(a)$$

$$+ \sum_{j=1}^{n} \left\{ \binom{n}{j-1} + \binom{n}{j} \right\} \left(f^{(j)}(a) g^{(n+1-j)}(a) \right)$$

Exercise (1.12h)

$$= f^{(n+1)}(a) g^{(0)}(a) + f^{(0)}(a) g^{(n+1)}(a)$$

$$+ \sum_{j=1}^{n} \binom{n+1}{j} f^{(j)}(a) g^{(n+1-j)}(a)$$

$$= \sum_{j=0}^{n+1} \binom{n+1}{j} f^{(j)}(a) g^{(n+1-j)}(a)$$

This is $\star$ with n replaced by $n+1$.

(29.12) (a) Let $f(x) = \tan x - x$ for $0 \le x < \pi/2$.

Then $f(0) = 0$ and $f'(x) = \sec^2 x - 1 > 0$ for $0 < x < \pi/2$.

By a result from class, f is strictly increasing on $[0, \pi/2)$. Thus for $0 < x < \pi/2$,

$f(x) = \tan x - x > 0$ ✓

(b) Let $g(x) = \dfrac{x}{\sin x}$, $0 < x < \frac{\pi}{2}$.

$$\text{Then } g'(x) = \frac{(\sin x) \cdot 1 - x \cos x}{\sin^2 x}$$

$$= \frac{\tan x - x}{\cos x \sin^2 x} > 0$$

By a result from class, g is strictly increasing on $(0, \frac{\pi}{2})$.

(c) Let $h(x) = \dfrac{x}{\sin x}$, $0 < x \le \pi/2$. By (b), h is strictly increasing on $(0, \frac{\pi}{2})$. Clearly h is continuous at $\frac{\pi}{2}$. Then $h(x) \le h(\frac{\pi}{2})$ for $0 < x \le \pi/2$. This says $\dfrac{x}{\sin x} \le \dfrac{\pi}{2}$ for $0 < x \le \frac{\pi}{2}$. Thus $x \le \frac{\pi}{2} \sin x$ for $0 \le x \le \frac{\pi}{2}$,

(29.17) (c) Suppose h is differentiable at a.

Then h is continuous at a, so $h(a) = \lim_{x \to a^-} h(x)$

$= \lim_{x \to a^-} f(x) = f(a)$ since f is continuous at a.

Now $h(a) = g(a)$ by definition, So $f(a) = g(a)$

We also have

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a^-} \frac{h(x) - h(a)}{x - a}$$

$$= \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = f'(a) \quad \text{and}$$

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a^+} \frac{h(x) - h(a)}{x - a}$$

$$= \lim_{x \to a^+} \frac{g(x) - g(a)}{x - a} = g'(a) \qquad \text{Thus } f'(a) = g'(a)$$

Now suppose $f(a) = g(a)$ and $f'(a) = g'(a)$. Then

$$\lim_{x \to a^+} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a^+} \frac{g(x) - g(a)}{x - a} = g'(a) \quad \text{and}$$

$$\lim_{x \to a^-} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Note we use $f(a) = g(a) = h(a)$ here.

Hence $h'(a) = g'(a) = f'(a)$