Math 445 Homework #2 Answers

4.6 a Suppose $x \in S$. Then, since $\sup S$ is an upper bound for S, we have $x \le \sup S$.

Likewise, since $\inf S$ is a lower bound for S, we have $\inf S \le x$. Combining, we get

$$\inf S \le x \le \sup S$$ ✓

4.6 (b) Suppose $\inf S = \sup S$. We claim this happens if and only if S consists of exactly one element.

(i) Suppose a is the only element of S. Then trivially $\sup S = a$ and $\inf S = a$, so

$$\inf S = \sup S.$$

(ii) Suppose S has two different elements. Call them a and b. WLOG $a < b$. Then we have

$$\inf S \le a < b \le \sup S,$$

so $\inf S \neq \sup S$ ✓

4.14a Suppose $x \in S$. Then $x = a+b$, where $a \in A$ and $b \in B$. Then $a \leq \sup A$ and $b \leq \sup B$. Hence $x = a+b \leq \sup A + \sup B$. Since $\sup A + \sup B$ is an upper bound for S, we conclude that

$$\sup S \leq \sup A + \sup B.$$

Now suppose

$$\sup S < \sup A + \sup B$$

and seek a contradiction.
We have $\sup A + \sup B - \sup S = 2\varepsilon > 0$.

Since $\sup A - \varepsilon < \sup A$, $\exists\, a \in A$ with $a > \sup A - \varepsilon$. Likewise there exists $b \in B$ with $b > \sup B - \varepsilon$. Then

$x = a+b \in S$ and

$$x > (\sup A - \varepsilon) + (\sup B - \varepsilon) = \sup S.$$

This is a contradiction, since $\sup S$ is an upper bound for S.