Math 447 Homework #7 Answers

(14.7) Since $\sum a_n$ converges, we know that $a_n \to 0$. Thus $\exists N \ni n > N \Rightarrow 0 \le a_n < 1$. Since $p > 1$, for $n > N$ we have $a_n^p < a_n$. We apply the Comparison Test to conclude that $\sum a_n^p$ converges.

(14.9) $\exists N \ni n > N \Rightarrow a_n = b_n$. Let

$$s_p = \sum_{n=1}^{p} a_n \quad \text{and} \quad t_p = \sum_{n=1}^{p} b_n.$$

Suppose $p > q > N$. Then

$$s_p - s_q = \sum_{n=q+1}^{p} a_n = \sum_{n=q+1}^{p} b_n = t_p - t_q.$$

Rearrange to obtain

$$s_p - t_p = s_q - t_q.$$

In other words, the sequence $s_p - t_p$ is constant for $p > N$. For $p > N$, write

$$s_p = t_p + c$$

Clearly $\lim_{p \to \infty} s_p$ exists if and only if $\lim_{p \to \infty} t_p$ exists. Equivalently, $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

(14c12) Since $\underline{\lim} |a_n| = 0$, $\exists$ subsequence a_{n_k} $\ni$ $|a_{n_k}| \to 0$ as $k \to \infty$. There in turns exists a further subsequence $a_{n_{k_j}}$ such that $|a_{n_{k_1}}| < 1$, $|a_{n_{k_2}}| < \frac{1}{2}$, $|a_{n_{k_3}}| < (\frac{1}{2})^2$, $|a_{n_{k_4}}| < (\frac{1}{2})^3$, and in general $|a_{n_{k_j}}| < (\frac{1}{2})^{j-1}$,

By the Comparison Test, $\sum_{j=1}^{\infty} a_{n_{k_j}}$ converges absolutely since $\sum_{j=1}^{\infty} (\frac{1}{2})^{j-1}$ converges.