Math 447 Homework #8 Answers

(17.13) (a) Suppose x is rational. For $n = 1, 2, 3, \ldots$,
let $x_n \in (x, x+\frac{1}{n})$ be irrational. Then $x_n \to x$,
We have $f(x_n) = 0$ for all n and $f(x) = 1$. Thus
$f(x_n) \not\to f(x)$. This shows f is discontinuous at x.
— A similar argument applies at irrational x's.

(b) Suppose x is irrational. There exists a
sequence x_n of rational x_n's converging to x.
Then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x \neq 0 = f(x)$.
Thus f is discontinuous at x.

Suppose x is rational and $x \neq 0$. Then
$\exists$ irrational $x_n \to x$. We have
$$\lim_{n\to\infty} f(x_n) = 0 \neq x = f(x)$$
Hence f is not continuous at x.

Finally suppose $x = 0$. Given $\varepsilon > 0$, set $\delta = \varepsilon$.
Then if $|t - x| < \delta$, we have
$|f(t) - f(x)| = |f(t)| < \delta = \varepsilon$. Then f is
continuous at $x = 0$.

(18.10) We follow the hint. Certainly g is continuous on $[0,1]$. Note $g(0) = f(1) - f(0)$ and $g(1) = f(2) - f(1)$ $= f(0) - f(1)$. We conclude, by the Intermediate Value Theorem, that there exists $x \in [0,1]$ such that $g(x) = 0$. For this x, $f(x+1) = f(x)$. Set $y = x+1$. ✓