Math 447 **Quiz # 3** **April 9, 2008**

NAME KEY

1. (12 points)

(i) State the Chain Rule. Suppose I and J are open intervals in R, $f: I \to J$, and $g: J \to R$. Let $h: I \to R$ be $h(x) = g(f(x))$. Suppose $a \in I$ and that f is differentiable at a and g is differentiable at $f(a)$. Then h is differentiable at a and

$$h'(a) = g'(f(a)) \cdot f'(a)$$

(ii) Let $f(x) = \begin{cases} x \sin \frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 . \end{cases}$

Does $f'(0)$ exist? Justify your answer. No.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \sin \frac{1}{x}$$

if this limit exists. But

$\lim_{x \to 0} \sin \frac{1}{x}$ does not exist; thus f is not differentiable at 0.

2. (12 points)

(i) State the Weierstrass M – test. Suppose $E \subset R$, $f_n : E \to R$, $M_n > 0$, and $\sum_{n=1}^{\infty} M_n < +\infty$. Suppose $\exists$ n_0 such that if $n > n_0$, then $|f_n(x)| \le M_n$ for all $x \in E$. Then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on E.

(ii) Show that $\sum_{n=1}^{\infty} \frac{x^n}{1+x^n}$ converges uniformly on (-1/2, 1/2).

Note that if $|x| < \frac{1}{2}$, then $\left| \frac{x^n}{1+x^n} \right| \le \frac{|x|^n}{1-|x|^n} \le \frac{(\frac{1}{2})^n}{1-\frac{1}{2}}$

$= (\frac{1}{2})^{n-1}$ for $n = 1, 2, 3, \ldots$

Since $\sum_{n=1}^{\infty} (\frac{1}{2})^{n-1}$ is a convergent geometric series, the Weierstrass M-test implies $\sum_{n=1}^{\infty} \frac{x^n}{1+x^n}$ converges uniformly on $(-\frac{1}{2}, \frac{1}{2})$.

3. (16 points) Consider the power series $\sum_{n=1}^{\infty} \frac{x^{2n}}{e^n}$. Let R be its radius of convergence.

(i) Find R.

$$R = \frac{1}{\overline{\lim}\left(\frac{1}{e^n}\right)^{\frac{1}{2n}}} = \frac{1}{\frac{1}{\sqrt{e}}} = \sqrt{e},$$

(Also note this is a geometric series $\sum_{n=1}^{\infty} \left(\frac{x^2}{e}\right)^n$,

and so it converges if $|x|^2 < e$ or $|x| < \sqrt{e}$.) ✓

(ii) Does this power series convergence uniformly on (-R, R) ? Justify your answer.

No. Let $F_N(x) = \sum_{n=1}^{N} \frac{x^{2n}}{e^n}$. Then for

every $N = 1, 2, 3, \ldots$,

$$\sup_{|x|<e^{1/2}} |F_{N+1}(x) - F_N(x)| = \sup_{|x|<\sqrt{e}} \left| \frac{x^{2(N+1)}}{e^{N+1}} \right| = 1.$$

Thus the sequence F_N is not uniformly Cauchy on $(-\sqrt{e}, \sqrt{e})$ and hence does not converge uniformly on $(-\sqrt{e}, \sqrt{e})$.