MATH 312, FALL 2002 - PROBLEM SET 11

WARMUP PROBLEMS: Section 5.1 #7. Section 5.2 #1, 2, 5. Do not write these up! Think about these to make sure you understand the material.

OTHER INTERESTING PROBLEMS: Section 5.1: #39, 49. Section 5.2 #7, 15, 16, 25, 26. Do not write up! Think about these if you have time.

WRITTEN PROBLEMS: Do five of the following six. Due Wednesday, Nov. 20.

1. Prove that $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$. (Hint: Use induction on $n(G)$.)

2. The results below imply that there is no k-critical graph with $k+1$ vertices.

a) Let x and y be vertices in a k-critical graph G. Prove that $N(x) \subseteq N(y)$ is impossible. Conclude that no k-critical graph has $k+1$ vertices.

b) Prove that $\chi(G \vee H) = \chi(G) + \chi(H)$, and that $G \vee H$ is color-critical if and only if both G and H are color-critical. Conclude that $C_5 \vee K_{k-3}$, with $k+2$ vertices, is k-critical.

3. Prove that if G is a color-critical graph, then the graph G' generated from it by applying Mycielski's construction is also color-critical.

4. *Turán's proof of Turán's Theorem.* Let $T_{n,r}$ denote the Turán graph with n vertices and r partite sets.

a) Prove that a maximal simple graph having no $(r+1)$-clique has an r-clique.

b) Prove that $e(T_{n,r}) = \binom{r}{2} + (n-r)(r-1) + e(T_{n-r,r})$.

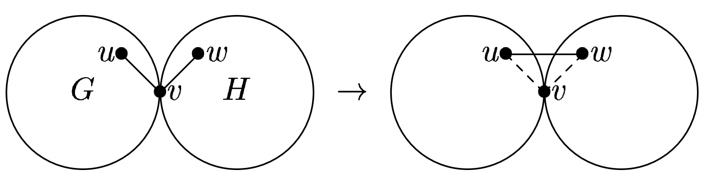
c) Use parts (a) and (b) to prove that $T_{n,r}$ has the most edges among all n-vertex graphs without $(r+1)$-cliques. (Comment: Turán also proved uniqueness of the extremal graph by this method.)

5. *The Hajós construction.*

a) Let G and H be k-critical graphs sharing only vertex v, with $vu \in E(G)$ and $vw \in E(H)$. Prove that the graph $(G - vu) \cup (H - vw) \cup uw$ also is k-critical.

b) For all $k \geq 3$, use part (a) to obtain a k-critical graph other than K_k.

c) For all $n \geq 4$ except $n = 5$, construct a 4-critical graph with n vertices.



6. Prove that every simple graph with minimum degree at least 3 contains a K_4-subdivision. (Hint: Prove a stronger result—every nontrivial simple graph with at most one vertex of degree less than 3 contains a K_4-subdivision. The proof of Theorem 5.2.20 already shows that every 3-connected graph contains a K_4-subdivision.)