MATH 312, FALL 2002 - PROBLEM SET 3

WARMUP PROBLEMS: Section 1.3: #7, 8. Section 1.4 #3, 4, 5, 6, 8, 19. Do not write up! Think about how to solve them to make sure you understand the material before doing the homework.

OTHER INTERESTING PROBLEMS: Section 1.3 #32, 41, 45, 52, 57, 61. Section 1.4 #10, 23, 24, 26, 29. Do not write up! Think about some of these if you have time.

WRITTEN PROBLEMS: Solve and write up five of the following six (students registered for one unit do all six). Due Wednesday, September 18.

1. Let G be an n-vertex simple graph, where $n \geq 2$. Determine the maximum possible number of edges in G under each of the following conditions.

a) G has an independent set of size a.

b) G has exactly k components.

c) G is disconnected.

2. Let G be a loopless graph with average vertex degree $a = 2e(G)/n(G)$.

a) Prove that $G - x$ has average degree at least a if and only if $d(x) \leq a/2$.

b) Use part (a) to give an algorithmic proof that if $a > 0$, then G has a subgraph with minimum degree greater than $a/2$.

c) Show that there is no constant c greater than $1/2$ such that G must have a subgraph with minimum degree greater than ca; this proves that the bound in part (b) is best possible. (Hint: Use $K_{1,n-1}$.)

3. Let $d_1, \ldots, d_n$ be integers such that $d_1 \geq \cdots \geq d_n \geq 0$. Prove that there is a loopless graph (multiple edges allowed) with degree sequence $d_1, \ldots, d_n$ if and only if $\sum d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$. (Hint: Any inductive proof of sufficiency must verify that the "smaller object" satisfies the condition before the induction hypothesis can be applied.)

4. Prove that in every digraph, some strong component has no entering edges, and some strong component has no exiting edges.

5. Let G be an n-vertex digraph with no cycles. Prove that the vertices of G can be ordered as $v_1, \ldots, v_n$ so that if $v_i v_j \in E(G)$, then $i < j$.

6. *DeBruijn sequence for any alphabet and length.* Let A be an alphabet of size k. Prove that there exists a cyclic arrangement of k^l characters chosen from A such that the k^l strings of length l in the sequence are all distinct.