MATH 312, FALL 2002 - PROBLEM SET 4

WARMUP PROBLEMS: Section 2.1: #2, 4, 8, 11, 12, 13. Do not write these up! Think about how to solve them to make sure you understand the material before doing the homework.

OTHER INTERESTING PROBLEMS: Section 1.4: #37, 38. Section 2.1: #21, 29, 31, 33, 35, 36, 44, 47. Do not write these up! Think about some of these if you have time.

WRITTEN HOMEWORK: Do five of the following six problems (students registered for one unit do all six). Due Wednesday, September 25.

1. By Proposition 1.4.30, every tournament has a king. Let T be a tournament having no vertex with indegree 0.

a) Prove that if x is a king in T, then T has another king in $N^-(x)$.

b) Use part (a) to prove that T has at least three kings.

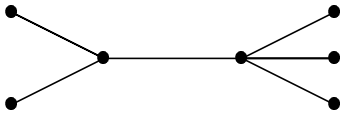
c) For each $n \geq 3$, construct an n-vertex tournament with exactly three kings.

(Comment: There exists an n-vertex tournament having exactly k kings whenever $n \geq k \geq 1$ except when $k = 2$ and when $n = k = 4$.)

2. Let $d_1, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum d_i = 2n - 2$. (Hint: Keep the quantifier in mind. Prove that the condition on the list of numbers is both necessary and sufficient for the *existence* of a tree with vertex degrees $d_1, \ldots, d_n$. Two implications must be proved.)

3. For $n \geq 3$, let G be an n-vertex graph such that every graph obtained by deleting one vertex is a tree. Determine $e(G)$, and use this to determine G itself.

4. Prove that the trees with diameter 3 are the *double-stars* (two central vertices plus leaves). Count the isomorphism classes of double-stars with n vertices.



5. For $n \geq 4$, prove that the minimum number of edges in an n-vertex graph with diameter 2 and maximum degree $n - 2$ is $2n - 4$.

6. Prove that $2k + 1$ is the maximum girth among all graphs with diameter k that are not trees. (Hint: Prove that if G has a cycle with length at least $2k + 2$, then G has a shorter cycle.)