MATH 475: Homework #2 (due 10/23/2002 at the end of the class)

Please do any three problems from (1)–(4) and any two problems from (5)–(7). Any additional problem will be counted towards future requirements.

(1) Suppose $n \geq 2$ and let $H = (V, E)$ be an n-uniform hypergraph with 4^{n-1} edges. Show that there is a coloring of V by four colors so that no edge is monochromatic.

(2) Prove that if there is a real p, $0 \leq p \leq 1$ such that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$

then the Ramsey number $R(k,t)$ satisfies $R(k,t) > n$. Using this show that

$$R(4,t) \geq \Omega(t^{3/2}/(\ln t)^{3/2}).$$

(3) Prove that there is a positive constant c so that every set A of n non-zero reals contains a subset $B \subset A$ of size $|B| \geq cn$ so that there are no $b_1, b_2, b_3, b_4 \in B$ satisfying

$$b_1 + 2b_2 = 2b_3 + 2b_4.$$

(4) Let $\mathbb{A} = (a_{ij})$ be an $n \times n$ matrix with all entries from $\{0, 1\}$. We call $\mathbb{A}$ *ninefree* if there is no 3×3 submatrix of all ones. (Note: the rows and columns of a submatrix needn't be consecutive) Let $f(n)$ denote the maximum number of ones in a ninefree $n \times n$ matrix. Give a precise theorem giving a lower bound for $f(n)$ and then analyze asymptotics, including the constant. *Hint:* consider a random matrix whose each entry is 1 is chosen independently with probability p. At the end you will need to optimize p.

(5) Prove that there is a constant $c > 0$ such that for every even $n \geq 4$ the following holds: for every undirected complete graph K on n vertices whose edges are colored red and blue, the number of alternating Hamilton cycles in K (i.e., properly edge-colored cycles of length n) is at most

$$n^c \frac{n!}{2^n}.$$

(6) Let $R_k(G)$ be the greatest integer n such that edges of K_n can be colored by k colors without monochromatic copy of G. Prove that

$$R_k(K_{3,3}) \geq \Omega(k^3/\log^3 k).$$

You may use the fact that there exist $K_{3,3}$-free graphs on n vertices with $n^{5/3}/2$ edges.

(7) Let X be a collection of pairwise orthogonal unit vectors in R^n and suppose the projection of each of these vectors on the the first k coordinates is of Euclidean norm at least ϵ. Show that $|X| \leq k/\epsilon^2$, and this is tight for all $\epsilon = k/2^r < 1$.