Homework #4 (due 12/11/2002)

Please do any three problems from (1)–(4) and any two problems from (5)–(7). Any additional problem will be counted towards future requirements.

(1) Prove (using Local Lemma) that Ramsey number $R(4,t)$ satisfies
$$R(4,t) \geq \Omega\big((t/\ln t)^{5/2}\big).$$

(2) A family of sets $\mathcal{G}$ is called *intersecting* if $G_1 \cap G_2 \neq \emptyset$ for all $G_1, G_2 \in \mathcal{G}$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be k intersecting families of subsets of $[n] = \{1, 2, \ldots, n\}$. Prove that
$$\Big|\bigcup_{i=1}^{k} \mathcal{F}_i\Big| \leq 2^n - 2^{n-k}.$$

(3) Let $G = (V, E)$ be a simple graph and suppose each $v \in V$ is associated with a set $S(v)$ of colors of size at least $10d$, ≥ 1. Suppose that for each $v \in V$ and $c \in S(v)$ there are at most d neighbors u of v such that $c \in S(u)$. Prove that there is a proper coloring of G assigning to each $v \in V$ a color from $S(v)$.

(4) Let G be a graph whose vertices are all 7^n vectors of length n over $\mathbb{Z}_7$, in which two vertices are adjacent iff they differ in precisely one coordinate. Let $U \subset V$ be a set of 7^{n-1} vertices of G, and let W be the set of all vertices of G whose distance from U exceeds $(c+2)\sqrt{n}$, where $C > 0$ is a constant. Prove that $|W| \leq 7^n e^{-c^2/2}$.

(5) The *Hajós number* of a graph G is the maximum number k such that there are k vertices in G with a path between each pair so that all the $\binom{k}{2}$ paths are internally pairwise disjoint (and no vertex is an internal vertex of a path and an endpoint of another). Is there a graph whose chromatic number exceeds twice its Hajós number ?

(6) Let G be a graph and let P denote the probability that a random subgraph of G obtained by picking each edge of G with probability $1/2$ (independently) is connected (and spanning). Let Q denote the probability that in a random two-coloring of edges of G, where each edge is chosen, randomly, independently, and uniformly, to be either red or blue, the red graph and the blue graph are both connected (and spanning). Is $Q \leq P^2$?

(7) Let $G = (v, E)$ be a graph with chromatic number $\chi(G) = 1000$. Let $U \subset V$ be a subset of V chosen uniformly among all $2^{|}V|$ subsets of V. Let $H = G[U]$ be the subgraph induced on U. Prove that
$$\text{Prob}(\chi(H) \leq 400) < 1/100.$$

Hint: prove first that the expectation of $\chi(H)$ is at least 500.