Cauchy-Schwarz and triangle inequalities

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The Cauchy-Schwarz inequality

The inequality, which holds in very general settings, is that

$$|\mathbf{v} \cdot \mathbf{w}| \leq ||\mathbf{v}||\ ||\mathbf{w}||. \tag{CS}$$

Note that the left-hand side of (CS) is the absolute value of the dot product, and the right-hand side is the product of their lengths. One way to see (CS) is to use the Law of Cosines, from which we get

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\ ||\mathbf{w}|| \cos(\theta).$$

If we know that $\mathbf{v} \cdot \mathbf{w} = ||v||\ ||w|| \cos(\theta)$, then the (CS) inequality says nothing more than that $-1 \leq \cos(\theta) \leq 1$.

On the other hand one can prove (CS) easily by using parametric equations for lines:

Proof. Notice that, if $\mathbf{w} = \mathbf{0}$, then both sides of (CS) are 0 and (CS) is true. Otherwise, if $\mathbf{w} \neq 0$, then $\mathbf{x} = \mathbf{v} + t\mathbf{v}$ gives parametric equations of a line through $\mathbf{v}$ with direction $\mathbf{w}$. We find the closest approach of this line to the origin by minimizing

$$f(t) = ||\mathbf{v} + t\mathbf{w}||^2 = ||\mathbf{v}||^2 + 2t\mathbf{v} \cdot \mathbf{w} + t^2 ||\mathbf{w}||^2. \tag{1}$$

Using calculus we set $f'(t) = 0$, getting

$$t = -\frac{v \cdot w}{||w||^2}. \tag{2}$$

Plug in this value of t in (1) to get

$$0 \leq ||v+tw||^2 = f(t) = ||v||^2 - 2\frac{|v \cdot w|^2}{||w||^2} + \frac{|v \cdot w|^2}{||w||^2} = ||v||^2 - \frac{|v \cdot w|^2}{||w||^2}.$$

Clearing out the $||w||^2$ gives the result

$$0 \leq ||v||^2 \; ||w||^2 - |v \cdot w|^2.$$

and we are done. □

Corollary 0.1 (the triangle inequality)**.**

$$||v + w|| \leq ||v|| + ||w|| \qquad (T).$$

Formula (T) states that the length of one side of a triangle is at most the sum of the other two sides; of course it is actually smaller for a non-degenerate triangle.

PROOF. The proof of (T) is immediate. (T) holds if and only if

$$||v + w||^2 \leq (||v|| + ||w||)^2$$

which holds if and only if

$$2v \cdot w \leq 2||v|| \; ||w||$$

which holds by (CS). □