DOT PRODUCT AND CROSS PRODUCT

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The algebra of vectors

What is a vector? In physics a vector is something with both a magnitude and a direction, such as velocity, the wind, the force of gravity, etc.

To make the notion precise, we define a vector to be a point in $\mathbf{R}^n$. Here $\mathbf{R}^n$ is the set of n-tuples of real numbers.

Thus a point in $\mathbf{R}^2$ is a pair (x, y) and a point in $\mathbf{R}^3$ is a triple (x, y, z). We sometimes think of the vector as the point, and sometimes we think of it as an arrow from the origin $(0, 0, 0)$ to the point. You need to be able to pass freely between these points of view. We will see many examples. The book uses the term *displacement vector* to help make the distinction.

Example 0.1. Consider the points $p = (2, 1)$ and $q = (1, 4)$ in the plane. We can think of each as an arrow from $(0, 0)$ to it. Then the vector $q - p = (-1, 3)$ is the displacement vector between p and q. Thus, starting at p, we add $q - p$ and we get to q. Notice that the vector $p - q$ is also a displacement vector, but it goes from q to p in the sense that $q + (p - q) = p$. There will be no ambiguity if you label the direction of the vector with an arrow.

Example 0.2. Parametric equations for lines in space

$$\mathbf{x}(t) = \mathbf{p} + t\mathbf{v}. \tag{0}$$

In other words

$$(x(t), y(t), z(t)) = (p_1, p_2, p_3) + t(a, b, c)$$

Here p is a point on the line and v is the direction vector of the line.

Remark 0.1. Equation (0) is the simplest example of the important idea of parametric equations. We think of $\mathbf{x}(t)$ as the position in $\mathbf{R}^3$ of a particle at time t. In (0), the particle moves along a line. In more interesting situations, such as the planets, the object moves along a more complicated curve, such as an ellipse. Notice that the *machine* that assigns to t the point $\mathbf{x}(t)$ is a function from $\mathbf{R}$ to $\mathbf{R}^3$.

Remark 0.2. In general, if $t \to \mathbf{x}(t)$ is a function from $\mathbf{R}$ to some set S, we think of $\mathbf{x}(t)$ as the position in S of a particle at time t. We don't always distinguish the function from the curve its values trace out. For example consider the function $t \to (\cos(t), \sin(t))$. For $0 \leq t \leq 2\pi$, the particle traces out a circle that starts at $(1, 0)$ and ends there.

Remark 0.3. If $t \to \mathbf{x}(t)$ is a differentiable function with values in $\mathbf{R}^3$, then its derivative $\mathbf{x}'(t)$ also has values in $\mathbf{R}^3$. It is called the velocity vector. Its magnitude (see below) is called the *speed*. For example, in equation (0), the derivative is $\mathbf{v}$.

Problems: Decide whether a point is on a line. Decide whether lines are parallel. Decide whether lines intersect, and if so, where. Find parametric equations for a line through two given points. We will discuss these concepts further later today or next time.

Two vectors are equal if and only if they have the same components:

Resolving a vector into components:

$$(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

Some Basic Definitions and Laws:

$$(x, y, z) + (a, b, c) = (x + a, y + b, z + c)$$

$$\lambda(x, y, z) = (\lambda x, \lambda y, \lambda z)$$

$$(x, y, z) \cdot (a, b, c) = xa + yb + zc$$

$$||v||^2 = v \cdot v$$

$$||(x, y, z)|| = \sqrt{x^2 + y^2 + z^2}$$

$$||\lambda v|| = |\lambda| \; ||v||$$

$$(x, y, z) \times (a, b, c) = (yc - bz, za - xc, xb - ay)$$

Both dot and cross products arise throughout physics and engineering. Dot products have to do with angles, projections, directional derivatives, *work*, etc. Cross products have to do with area, *torque*, the Lorentz force law, etc. Both products are indispensable to major parts of this course.

HOW DO WE REMEMBER the formula for cross product? We use determinants as follows:

$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a & b & c \end{pmatrix} =$$

$$(yc-bz)\mathbf{i}+(za-xc)\mathbf{j}+(xb-ay)\mathbf{k} = (yc-bz, za-xc, xb-ay).$$

Meaning of addition is easy.

Meaning of scalar multiplication is easy.

Meaning of dot product.

Theorem 0.1. *Relationship between dot product and angles:*

$$v \cdot w = ||v||\ ||w||\ \cos(\theta). \tag{1}$$

PROOF. We use the law of cosines

$$||v||^2 - 2v \cdot w + ||w||^2 = ||v - w||^2 = ||v||^2 + ||w||^2 - 2||v||\ ||w|| \cos(\theta)$$

and the result (1) follows. □

Theorem 0.2. *Relationship between cross product and angles:*

$$||v \times w|| = ||v||\ ||w||\ \sin(\theta). \tag{2}$$

PROOF. We use the previous item and the following formula:

$$||v \times w||^2 + |v \cdot w|^2 = ||v||^2 ||w||^2.$$

This formula is easily proved by plugging in the definitions. Can you find a better proof? We get

$$||v \times w||^2 = ||v||^2 ||w||^2 (1 - \cos^2(\theta))$$

from which the rule (2) follows. □

Remark 0.4. In higher math, one often angles via formula (1) and then the law of cosines follows as a theorem.

Remark 0.5. The pythagorean theorem and the law of cosines are equivalent. Each implies the other.

Corollary 0.1. *Vectors are perpendicular (orthogonal) if and only if their dot product is* 0.

Projections are very important in science and math. Consider the stupid projector in this room!

$$\text{proj}_{\mathbf{w}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||^2} \mathbf{w}$$

The answer must be a multiple of $\mathbf{w}$. It must be $\mathbf{w}$ when $\mathbf{v} = \mathbf{w}$, and it must be 0 when $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

Cross products are amusing. Note that $v \times w = -w \times v$. The cross products of the standard basis vectors satisfy:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$
$$\mathbf{j} \times \mathbf{k} = \mathbf{i}$$
$$\mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Remark 0.6. The area of a parallelogram or triangle in space is defined via the cross product. Let $\mathcal{P}$ be the parallelogram spanned by the vectors v and w. Then the area of $\mathcal{P}$ is given by

$$A(\mathcal{P}) = ||v \times w||.$$

Example 0.3. Find the area of the triangle through $(1, 2, 3)$, $(2, 5, 6)$, and $(2, 2, 3)$.

$$A = \frac{1}{2}||(1, 3, 3) \times (1, 0, 0)|| = \frac{1}{2}||(0, 3, -3)|| = \frac{\sqrt{18}}{2}.$$

What would happen if you tried to find the area of a triangle as above when the three points were collinear?

In doing Kepler's laws, we will use the following weird identity for the cross product of vectors in $\mathbf{R}^3$. Here a, b, c are vectors

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \qquad (1)$$

To verify (1) we first note that both sides are linear in $\mathbf{a}$. It suffices therefore to prove it when $\mathbf{a} = (1, 0, 0)$. Then $\mathbf{a} \cdot \mathbf{b} = b_1$ and $\mathbf{a} \cdot \mathbf{c} = c_1$ and the identity becomes

$$(0, b_2c_1 - b_1c_2, -b_1c_3 + b_3c_1) = c_1(0, b_2, b_3) + b_1(0, -c_2, -c3) = c_1\mathbf{b} - b_1\mathbf{c},$$

which is true. (Notice that the first component of $c_1\mathbf{b} - b_1\mathbf{c}$ is 0.)

The triple scalar product. Sometimes one considers the number

$$u \cdot (v \times w).$$

This number is called the triple scalar product of u, v, w and it equals the oriented volume of the three-dimensional box spanned by them. It equals the determinant of the three by three matrix whose rows are u, v, w.