Lines and Planes

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Lines and Planes in $\mathbf{R}^n$

lines

The best way to describe a line is through parametric equations. Thus we let $r(t)$ denote the position along the line at time t. The line is then described as the set of points in $\mathbf{R}^n$ of the form $r(t)$ as $-\infty < t < \infty$.

If we know a point on a line, and we know the direction of the line, then we know the line. Therefore we have

$$r(t) = \mathbf{p} + t\mathbf{v},$$

where $\mathbf{p}$ is a point on the line, $\mathbf{v}$ is the direction of the line, and t is a real parameter. Since $r(t)$ is the position of a particle at time t, when $t = 0$ we are at $\mathbf{p}$. Notice that the parametric equations are not unique. We can use a different initial point, and we can multiply $\mathbf{v}$ by a nonzero constant.

Example: Find the equation of a line through $(1, 2, 3)$ and $(3, 0, 9)$. By subtracting the two points we get the displacement vector $(2, -2, 6)$ which defines the direction of the line. Therefore we can use

$r(t) = (x(t), y(t), z(t)) = (1, 2, 3) + t(2, -2, 6).$

We could have used any of the following

$$(1, 2, 3) + t(4, -4, 12)$$
$$(3, 0, 9) + t(-10, 10, -30)$$
$$(1, 2, 3) + t(-4, 4, -12)$$

for example.

Two lines are *parallel* if the have the same direction but do not intersect. Thus their direction vectors must be non-zero multiples.

Example: the line given by $(1 + t, 2 + 3t, 7)$ is parallel to the line given by $(2s, 6s, 7)$ but the lines given by $(1 + t, 2 + 3t, 7)$ and $(2s + 2, 5 + 6s, 7)$ are the same line!

Remark: Later on we will consider curves gievn by parametric equations. For example $(x(t), y(t), z(t)) = (\cos(t), \sin(t), t)$ defines a helix. It is crucial to understand the special case of lines, because infinitesimally a smooth curve behaves like its tangent line.

Remark: Sometimes it is useful to think of a line as the intersection of two planes. For example the sets $3x + y = 8$ and $x + y + z = 6$ define planes. Where do they intersect? We have the two equations $3x + y = 8$ and $x + y = 6 - z$, hence $x = 1 + \frac{z}{2}$ and $y = 5 - \frac{3z}{2}$. Take $z = 2t$ and we get

$$(x(t), y(t), z(t)) = (1 + t, 5 - 3t, 2t).$$

Check: $3(1+t)+(5-3t)=8$ checks, and $(1+t)+(5-3t)+2t=6$ checks.

Example 0.1. Give an example of skew lines, and justify your answer. Consider the line $(x,y,z)=t(1,0,0)$. To find another line which doesn't intersect this line and which is not parallel, try any other direction, say $(0,1,0)$. Then consider the line

$$(a,b,c)+s(0,1,0).$$

(Notice that I used a different parameter. Why?) Thus we need

$$(a,b+s,c)=(t,0,0)$$

to have no solutions. Pick any c different from 0. Then the lines do not intersect.

Sometimes you will see so-called *symmetric equations* for a line. The idea is a bit dumb, as it amounts to rewriting the parametric equations. Example, consider the line given by $(1+2t,3-t,2t)$. We can solve for t in three ways to get

$$t=\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z}{2}.$$

Hence sometimes one will see the line defined by

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z}{2}$$

without explicit reference to t. In this way we are also writing the line as an intersection of two planes.

planes

The best way to describe a plane in $\mathbf{R}^3$ is through a defining equation. We need to know the normal vector to the plane and one point on the plane. Thus $\mathcal{P}$ is given by

$$(\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = 0.$$

In this version, $\mathbf{p}$ is on the plane, and $\mathbf{n}$ is the normal vector. Equivalently we could write

$$\mathbf{x} \cdot \mathbf{n} = D$$

for some constant D, which has to be $\mathbf{p} \cdot \mathbf{n}$.

Example. Find a plane through the points $(1, 2, 3)$, $(3, 4, 5)$, $(1, 0, -1)$. Decide whether this plane contains the origin and also the point $(2, 4, 1)$.

Method 1: Find two displacement vectors in the plane. Take their cross product to find $\mathbf{n}$. Here the displacement vectors are $(2, 2, 2)$ and $(0, -2, -4)$. Their cross product is $(-4, 8, -4)$. We could use this vector as $\mathbf{n}$, or we could use any multiple, say $(1, -2, 1)$. Thus a defining equation for the plane $\mathcal{P}$ is given by

$$1(x - 1) - 2(y - 2) + 1(z - 3) = 0,$$

which is more nicely written

$$x - 2y + z = 0.$$

Thus the origin lies on the plane, but the point $(1, -2, 1)$ does not as $1 + 4 + 1 \neq 0$.

Method 2: We know the plane has an equation

$$Ax + By + Cz = d.$$

We thus have the equations $A + 2B + 3C = D$, $3A + 4B + 2C = D$, and $A - C = D$. We can make these equations into a linear system. We can solve these by the usual methods.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} D \\ D \\ D \end{pmatrix}$$

Doing row operations we get

$$\begin{pmatrix} 1 & 2 & 3 & D \\ 3 & 4 & 2 & D \\ 1 & 0 & -1 & D \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & D \\ 0 & -2 & -4 & -2D \\ 0 & -2 & -4 & 0 \end{pmatrix}$$

Hence we have $D = 0$ and also

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -2 & -4 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

From the last matrix we can take $A = C = 1$ and $B = -2$.

Example 0.2. What is the intersection of the line given by $(1 + t, 2 - t, 3)$ with the plane given by

$x+y+z=6$? With the plane $x-z=0$? We plug the parametric equations into the defining equation. In the first case we get:

$$(1+t)+(2-t)+3=6,$$

which holds for all t, and hence the line is on the plane.

In the second case we get

$$(1+t)-3=0,$$

which holds if and only if $t=2$. Thus the point $(3,0,3)$ is the only point on both the line and the plane.

We can also describe planes by parametric equations, but we need two parameters. For example the plane $3x+2y+z=14$ can be described as the image of the map

$$(u,v)\to(u,v,14-3u-2v).$$

These ideas will be fundamental later when we do surface integrals to find surface area, electric flux, etc. We will then have curved surfaces given by parametric equations. It is crucial to understand the special case of planes, because infinitesimally a smooth surface behaves like its tangent plane.

Here is another way to describe a plane. Choose two independent vectors $\mathbf{v}$ and $\mathbf{w}$. Let $\mathcal{P}$ be the plane they span; thus

$$\mathcal{P} = \{\mathbf{u} : \mathbf{u} = s\mathbf{v} + t\mathbf{w}\},$$

where s, t range over all real numbers. This method is useful because it gives parametric equations for things such as parallelograms and triangles.

distance problems

Example 0.3. Find the distance between parallel planes. Consider $\mathbf{x} \cdot \mathbf{n} = D_1$ and $\mathbf{x} \cdot \mathbf{n} = D_2$. What is the distance between them? Take points v_1 on P_1 and v_2 on P_2 and project their difference onto $\mathbf{n}$.

$$\mathrm{proj}_{\mathbf{n}}(v_1 - v_2) = \frac{(v_2 - v_1) \cdot n}{||\mathbf{n}||^2}\mathbf{n}$$

and hence the length is

$$\frac{|(v_2 - v_1) \cdot \mathbf{n}|}{||\mathbf{n}||} = \frac{|D_2 - D_1|}{||\mathbf{n}||}.$$

Using the above formula, the distance between the parallel planes given by $x+y+z = 1$ and $x+y+z = 0$ is $\frac{1}{\sqrt{3}}$.

Example 0.4. Find the distance between a point and a line. Suppose the point is $(2, 1, 3)$ and the line is given by $(2 - t, 3 + t, -2 - 2t)$. There are many methods.

1) Distance squared is $\delta(t) = t^2 + (2 + t)^2 + (1 - 2t)^2 = 6t^2 + 5$, which obviously has a minimum of 5 when $t = 0$. Hence the distance is $\sqrt{5}$.

2) We want the height of the right triangle formed by the points $(2,1,3)$, $(2,3,-2)$ and $(2-t, 3+t, -2-2t)$ for the right value of t. So we project the hypotenuse $(0,-2,5)$ on the direction $\mathbf{v} = (-1,1,-2)$ and then use the Pythagorean theorem. The side in the line is then given by

$$\frac{(0,-2,5)\cdot(-1,1,-2)}{||\mathbf{v}||^2}\mathbf{v} = \frac{-12}{6}\mathbf{v} = (2,-2,4).$$

We thus have a right triangle with side lengths $\sqrt{29}$ and $\sqrt{24}$. The third length is therefore $\sqrt{5}$.

3) We can use cross products. We know that

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}},$$

and we can find $\sin(\theta)$ using cross products.

As above the hypotenuse is the vector $(0,-2,5)$. The vector $\mathbf{v}$ is still $(-1,1,-2)$. Their cross product is given by $(1,5,2)$ and hence $\sqrt{30} = \sqrt{6}\sqrt{29}\sin(\theta)$

$$\text{opposite} = \sqrt{29}\sin(\theta) = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

Method 3) leads to a fairly simple formula. Don't memorize it, but be able to derive it:

Distance from p to the line $q + t\mathbf{v}$ is given by

$$\text{distance} = \frac{||(p-q)\times\mathbf{v}||}{||\mathbf{v}||}. \tag{11}$$

Method 2) leads to the formula

$$\text{distance}^2 = ||p - q||^2 - \frac{||(p - q) \times \mathbf{v}||^2}{||\mathbf{v}||^2}. \qquad (12)$$

Formulas (11) and (12) are easily seen to be the same.

Finding the distance between parallel lines is similar to finding the distance between parallel planes, but you need to find the right normal vector!

Find the distance between a line and a plane. Here one would expect the line to intersect the plane, in which case the distance is zero. But, it could be that the line is parallel to the plane. In this case the direction of the line is orthogonal to the normal of the plane. Then we can reduce the problem to finding the distance between parallel planes.

Find the distance between skew lines. Again, the lines must lie in parallel planes, and again we reduce the problem to finding the distance between parallel planes.