Skills

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operations on vectors

1) adding vectors
2) scalar multiplication
3) length of a vector
4) dot product, relation to angles,
5) cross product, relation to angles, right hand rule
6) notions of perpendicular and parallel
7) projection of v onto $\mathbf{n}$:

$$\frac{v \cdot n}{||\mathbf{n}||^2}\mathbf{n}$$

Note that the length of the projection is

$$\frac{|v \cdot n|}{||\mathbf{n}||}.$$

8) areas of parallelograms and triangles in space. Consider parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$:

$$\text{Area(parallelogram)} = ||\mathbf{v} \times \mathbf{w}||$$

lines and planes

1) parametric and symmetric equations for lines

$$x(t) = \mathbf{p} + t\mathbf{v}$$

2) point-normal equation for a plane

$$(\mathbf{x}-\mathbf{p})\cdot\mathbf{n}=0.$$

3) intersections of objects
4) distance problems

distance problems

Remark 0.1. In distance problems, the word *distance* means the minimum distance. If, for example, a point is on a line or plane, then the distance to the line or plane is 0. In principle, one can find distances by calculus; write down a formula for the distance between arbitrary appropriate points and use calculus to minimize it. In practice, most of the time it is easier to use geometric methods. Below we give some examples.

Here are some of the sorts you things you should be able to do. DO NOT MEMORIZE FORMULAS.
1) distance from a point to a plane
2) distance between parallel planes
3) distance from a line to a plane
4) distance between two skew lines

Example 0.1. Find the distance between parallel planes. Consider $\mathbf{x}\cdot\mathbf{n}=D_1$ and $\mathbf{x}\cdot\mathbf{n}=D_2$. What is the distance between them? Take points v_1 on P_1 and v_2 on P_2 and project their difference onto $\mathbf{n}$.

$$\text{proj}_{\mathbf{n}}(v_1-v_2)=\frac{(v_2-v_1)\cdot n}{||\mathbf{n}||^2}\mathbf{n}$$

and hence the length is

$$\frac{|(v_2 - v_1) \cdot \mathbf{n}|}{||\mathbf{n}||} = \frac{|D_2 - D_1|}{||\mathbf{n}||}. \tag{1}$$

Example 0.2. Find the distance between the parallel planes $x+2y+4z = 17$ and $x+2y+4z = 7$. Note that $\mathbf{n} = (1, 2, 4)$. Note for example that $(17, 0, 0)$ is on the first plane and $(7, 0, 0)$ is on the second plane. A displacement vector between them is $(10, 0, 0)$; we take the length of the projection of this vector onto $\mathbf{n}$ to get

$$\text{distance} = \frac{|(10, 0, 0) \cdot (1, 2, 4)|}{||(1, 2, 4)||} = \frac{10}{\sqrt{21}}. \tag{2}$$

Note that we could have chosen different points. For example $(1, 0, 4)$ is on the first plane and $(1, 1, 1)$ is on the second. Using these we get the displacement vector $(0, 1, -3)$. Now we get:

$$\text{distance} = \frac{|(0, 1, -3) \cdot (1, 2, 4)|}{||(1, 2, 4)||} = \frac{10}{\sqrt{21}}. \tag{3}$$

Note in the previous example that $|(v_2 - v_1) \cdot \mathbf{n}|$ will always be $10 = 17 - 7$, as noted in the proof of (1).

Example 0.3. Find the distance between a point and a line. Suppose the point is $(2, 1, 3)$ and the line is given by $(2 - t, 3 + t, -2 - 2t)$. There are many methods.

1) Distance squared is $\delta(t) = t^2 + (2+t)^2 + (1-2t)^2 = 6t^2 + 5$, which obviously has a minimum of 5 when $t = 0$. Hence the distance is $\sqrt{5}$.

2) We want the height of the right triangle formed by the points $(2,1,3)$, $(2,3,-2)$ and $(2-t, 3+t, -2-2t)$ for the right value of t. So we project the hypotenuse $(0,-2,5)$ on the direction $\mathbf{v} = (-1,1,-2)$ and then use the Pythagorean theorem. The side in the line is then given by

$$\frac{(0,-2,5)\cdot(-1,1,-2)}{||\mathbf{v}||^2}\mathbf{v} = \frac{-12}{6}\mathbf{v} = (2,-2,4).$$

We thus have a right triangle with side lengths $\sqrt{29}$ and $\sqrt{24}$. The third length is therefore $\sqrt{5}$.

3) We can use cross products. We know that

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}},$$

and we can find $\sin(\theta)$ using cross products.

As above the hypotenuse is the vector $(0,-2,5)$. The vector $\mathbf{v}$ is still $(-1,1,-2)$. Their cross product is given by $(1,5,2)$ and hence $\sqrt{30} = \sqrt{6}\sqrt{29}\sin(\theta)$

$$\text{opposite} = \sqrt{29}\sin(\theta) = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

Method 3) leads to a fairly simple formula. Don't memorize it, but be able to derive it:

Distance from p to the line $q + t\mathbf{v}$ is given by

$$\text{distance} = \frac{||(p-q)\times \mathbf{v}||}{||\mathbf{v}||}. \tag{11}$$

Method 2) leads to the formula

$$\text{distance}^2 = ||p-q||^2 - \frac{||(p-q)\times \mathbf{v}||^2}{||\mathbf{v}||^2}. \tag{12}$$

Formulas (11) and (12) are easily seen to be the same.

Finding the distance between parallel lines is similar to finding the distance between parallel planes, but you need to find the right normal vector!

Example 0.4. Consider the lines $(x, y, z) = (1, 2, 3) + t(2, 0, 4)$ and $(x, y, z) = (1, 0, 0) + s(1, 0, 2)$. They have the same direction vector. Do they intersect? If not, they must be parallel. What then is the distance between them?

Do they intersect?

$$(1 + 2t, 2, 3 + 4t) = (1 + s, 0, 2s)$$

has no solutions (by inspection of the **j** component). Hence they do not intersect. Thus they are parallel. How do we find the distance between them?

Here is one of several similar ways to go. We want to find s such that the point $(1+s, 0, 2s)$ is closest to the point $(1, 2, 3)$. To do so, we want their difference to be perpendicular to the direction vector $(2, 0, 4)$:

$$0 = ((1, 2, 3) - (1 + s, 0, 2s)) \cdot (2, 0, 4) =$$

$(-s, 2, 3-2s)\cdot(2, 0, 4) = -2s+12-8s = 12-10s$. Thus we put $s = \frac{6}{5}$ and the point is $(11/5, 0, 12/5)$. The distance between this point and $(1, 2, 3)$ is

$$\sqrt{\frac{36}{25}+4+\frac{9}{25}} = \sqrt{\frac{145}{25}} = \sqrt{\frac{29}{5}}.$$

Equivalently we could find the distance from $(1+s, 0, 2s)$ to $(1, 2, 3)$ and minimize it. The squared distancce (always easier to work with the squared distance) is

$$s^2 + 4 + (2s-3)^2 = f(s)$$

and again we can use calculus to see that f is minimized when $s = \frac{6}{5}$.

Find the distance between a line and a plane. Here one would expect the line to intersect the plane, in which case the distance is zero. But, it could be that the line is parallel to the plane. In this case the direction of the line is orthogonal to the normal of the plane. Then we can reduce the problem to finding the distance between parallel planes.

Find the distance between skew lines. Again, the lines must lie in parallel planes, and again we reduce the problem to finding the distance between parallel planes.