Smooth curves

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Formulas for particles moving along curves

Let $\gamma : [0,1] \to \mathbf{R}^3$ be a smooth (at least twice differentiable) curve. Here are things we need to know.

1) $\gamma(t)$ represents the position of a moving particle at time t.

2) $\gamma'(t)$ represents its velocity vector at time t. We often write $v(t)$. It is a vector.

3) $||\gamma'(t)|| = ||v(t)||$ is the speed at time t. It is a scalar.

4) $\gamma''(t)$ represents its acceleration vector at time t. We often write $a(t)$. It is a vector.

5) The unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{v(t)}{||v(t)||}.$$

6) The arc-length of the curve from 0 to t is given by

$$L_\gamma = \int_0^t ||\gamma'(\tau)||d\tau.$$

7) The arc-length parameter is given by

$$s = s(t) = \int_0^t ||\gamma'(\tau)||d\tau.$$

We have, by the fundamental theorem of calculus,

$$\frac{ds}{dt} = ||\gamma'(t)|| = ||v(t)||.$$

8) Let $\mathbf{N}(t)$ be the unit normal; it is perpendicular to $\mathbf{T}(t)$ and in the plane spanned by $v(t)$ and $a(t)$. We have the crucial formula defining curvature.

$$a(t) = \frac{d}{dt}(||v(t)||)\mathbf{T}(t) + K(t)||v(t)||^2\mathbf{N}(t). \quad (5)$$

We rewrite this formula dropping the variable t to make it easier to look at:

$$a = \frac{d}{dt}(||v||)\mathbf{T} + K||v||^2\mathbf{N}. \quad (6)$$

9) It follows from (6) that

$$K = \frac{||v \times a||}{||v||^3}, \quad (7)$$

the easiest formula to use in practice. One also gets

$$K = \frac{a \cdot \mathbf{N}}{||v||^2}.$$

10) The book defines K another way. One uses the arc-length parameter s. Here is a simple way to see that the two definitions are the same.

We start with

$$v = ||v||\mathbf{T}.$$

Differentiating gives

$$a = \frac{d}{dt}(||v||)\mathbf{T} + ||v||\mathbf{T}'. \tag{8}$$

The first term is what we had. We need to look at the second. Use the arc-length parameter and (8) to write

$$a = \frac{d}{dt}(||v||)\mathbf{T} + ||v||^2\frac{d\mathbf{T}}{ds}. \tag{9}$$

Comparing (9) with (6), and knowing that $\mathbf{N}$ is a unit vector, shows that

$$K = ||\frac{d\mathbf{T}}{ds}||.$$

For a helix, for example, the curvature is constant, and it is instructive to compute it by both formulas. Doing so will help you understand the arc-length parameter s.

11) Last, and probably least, one defines $\mathbf{B}$ by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.

Then the triple $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ are what is called a *moving frame*. (Comes from the French; developed by Eli Cartan) The idea is that, while these three vectors change from point to point, they are mutually orthogonal and of unit length at all points. We get an analogue of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ adapted to the curve.