

Kepler's Laws

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We will use the following weird identity for the cross product of vectors in \mathbf{R}^3 :

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c. \quad (1)$$

To verify (1) we first note that both sides are linear in a . It suffices therefore to prove it when $a = (1, 0, 0)$. Then $a \cdot b = b_1$ and $a \cdot c = c_1$ and the identity becomes

$$(0, b_2c_1 - b_1c_2, -b_1c_3 + b_3c_1) = c_1(0, b_2, b_3) + b_1(0, -c_2, -c_3) = c_1b - b_1c,$$

which is true.

If we apply (1) to $a = b = r$ and $c = v$ we get

$$r \times (r \times v) = r \cdot v r - r \cdot r v. \quad (2)$$

Let $r(t)$ denote the position of the planet at time t . As usual $v = r'$ denotes the velocity and $a = v'$ denotes the acceleration. We assume that the sun is at the origin and that M is the mass of the sun. Let m be the mass of the planet. Newton's law of gravitation gives

$$ma = F = -GMm \frac{r}{\|r\|^3}.$$

Therefore

$$a = r'' = -GM \frac{r}{\|r\|^3}.$$

Lemma 0.1. *The motion is in a plane.*

PROOF. The vector $r \times v$ is normal to the motion, because of the definition of cross product. Call it c . If we show that c is a (non-zero) constant vector, then it defines the normal to the plane of motion. (If $c = 0$, then the planet is moving in a line and hence in a plane as well.) But

$$c' = (r \times v)' = v \times v + r \times a = r \times a = 0,$$

because a is a multiple of r . □

Lemma 0.2. $\|r\|' = \frac{r \cdot v}{\|r\|}$

PROOF. We have

$$\frac{d}{dt}\|r\| = \left(\frac{d}{dt}\right)(\|r\|^2)^{\frac{1}{2}} = \frac{1}{2\|r\|} \left(\frac{d}{dt}\right)(r \cdot r) = \frac{r \cdot v}{\|r\|}$$

□

Put $c = r \times v$. From Lemma 1, c is a constant vector.

Lemma 0.3. *The vector $v \times c - GM \frac{r}{\|r\|}$ is a constant.*

PROOF. Call this vector d , differentiate, and use the Lemmas to get

$$d' = v' \times c - GM \left(\frac{r}{\|r\|}\right)'$$

$$\begin{aligned}
&= -GM \frac{r}{\|r\|^3} \times (r \times v) - GM \left(\frac{\|r\|r' - r\|r\|'}{\|r\|^2} \right) \\
&= -\frac{GM}{\|r\|^3} (r \times (r \times v) + \|r\|^2 r' - r r \cdot r') \\
&= -\frac{GM}{\|r\|^3} (r \cdot v r - r \cdot r v + r \cdot r v - r r \cdot v) = 0.
\end{aligned}$$

Notice in the last step that we used (2) above. \square

From Lemma 3 we obtain the formula

$$d = v \times c - GM \frac{r}{\|r\|^2}.$$

Now we compute

$$\|c\|^2 = c \cdot c = (r \times c) \cdot c = (d + GM \frac{r}{\|r\|^2}) \cdot c = \|r\| (GM + \|d\| \cos(\theta)),$$

from which we obtain the equation

$$\|r\| = \frac{\|c\|^2}{GM + \|d\| \cos(\theta)}. \quad (10)$$

Equation (10) is a polar equation, whose locus is an ellipse, parabola, or hyperbola. We cannot eliminate the last two possibilities, because in fact they can occur. If the orbit is closed, however, the orbit must be an ellipse.

Lemma 0.4. *The polar equation $\rho = \frac{\lambda}{\mu + \nu \cos(\theta)}$ is an ellipse if $\mu^2 > \nu^2$.*

PROOF. Rewrite the equation as

$$\rho(\mu + \nu \cos(\theta)) = \lambda.$$

Replace ρ by $\sqrt{x^2 + y^2}$ and replace $\rho \cos(\theta)$ by x to get

$$\mu \sqrt{x^2 + y^2} + \nu x = \lambda$$

$$\mu^2(x^2 + y^2) = (\lambda - \nu x)^2$$

which is the equation of a conic section. It defines an ellipse if $\mu^2 > \nu^2$. Using the constants arising from (10) we see that the condition for being an ellipse is that $GM > ||d||$ \square

First Law

The motion is an ellipse.

Second Law

Equal areas are swept out in equal time intervals.

The area swept out as θ varies from θ_1 to θ_2 is the integral

$$A = \int_{\theta_1}^{\theta_2} \frac{r^2}{2} d\theta.$$

We can calculate the rate of change of the area with time by using the chain rule:

$$\frac{dA}{dt} = \frac{dA}{d\theta} \frac{d\theta}{dt} = \frac{r^2}{2} \frac{d\theta}{dt}. \quad (11)$$

To show that (11) is a constant we need to find $\frac{d\theta}{dt}$.

Lemma 0.5. $\|c\| = \|r\|^2 \frac{d\theta}{dt}$.

PROOF. We have

$$c = r \times v = \|r\|(\cos(\theta), \sin(\theta), 0) \times v.$$

Since $v = r'$, it follows that

$$v = \|r\|'(\cos(\theta), \sin(\theta), 0) + \|r\| \frac{d\theta}{dt}(-\sin(\theta), \cos(\theta), 0)$$

and hence that

$$c = r \times v = \|r\|^2 \frac{d\theta}{dt}(0, 0, 1).$$

Therefore $\|c\| = \|r\|^2 \frac{d\theta}{dt}$. □

We therefore discover that $\frac{dA}{dt} = \frac{\|c\|}{2}$. Since this value is independent of t , we conclude Kepler's second law.

Third Law

The squared length T^2 of time for one orbit is proportional to the cube of the (semi)-major axis of the ellipse of motion. The area of the ellipse is given by πab . Hence

$$\pi ab = \int_0^T \frac{dA}{dt} dt = \int_0^T \frac{\|c\|}{2} dt = T \frac{\|c\|}{2}.$$

Thus T^2 is a constant times $a^2 b^2$. Using the equation of the ellipse to eliminate b , we discover that b^2 is a constant times a , and the result follows.