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Math 225, Spring 2010
Midterm IIB
March 18, 2010

NAME KEY

A) DO NOT OPEN YOUR EXAM UNTIL YOU ARE TOLD.

B) Be sure to write your name on this page.

C) This is a closed book, closed notes exam. No calculators allowed. Show all of your work or complete credit may not be given.

D) No cell phones allowed. Please turn them OFF during the exam.

E) No other electronic devices allowed e.g. MP3 players, PDAs, etc.

F) If you have a question, raise your hand and the proctor will come to you.

G) If you finish early, please hand in your exam and leave quietly in consideration of your fellow students. You will need to show a picture ID with a clear picture when you turn in your exam. When time is up, you will be instructed to put down your writing utensil and close your exam.

H) Every exam is worth a total of 100 points. Each exam has 6 pages including the cover sheet. Check to see that you have all the pages.

1) (20pts) Consider the 2×2 matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}, C = \begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix}.$$

a) Compute $D = AB^{-1}C$.
b) Show by a direct computation that $\det D = \frac{\det A \det C}{\det B}$.

$$D = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \left(\frac{1}{-1} \begin{bmatrix} 1 & -1 \\ -4 & 3 \end{bmatrix} \right) \begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+12 & 1-9 \\ -2+4 & 2-3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -38 & 3 \\ -6 & 1 \end{bmatrix}$$

$$\det D = -38 + 18 = -20$$

$$\frac{\det A \det C}{\det B} = \frac{-5 \quad -4}{-1} = -20$$

2) (20pts) Let a, b, c be real numbers. Consider the 3×3 matrix

$$A = \begin{bmatrix} a & 2 & 1 \\ b & 3 & 3 \\ c & 4 & 5 \end{bmatrix}.$$

a) What is the formula for the determinant of A in terms of a, b, c?
b) Give an example of a, b, c for which A is not invertible and give an example of a, b, c for which A is invertible.

$$\det A = a \begin{vmatrix} 3 & 3 \\ 4 & 5 \end{vmatrix} - b \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix} + c \begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix}$$

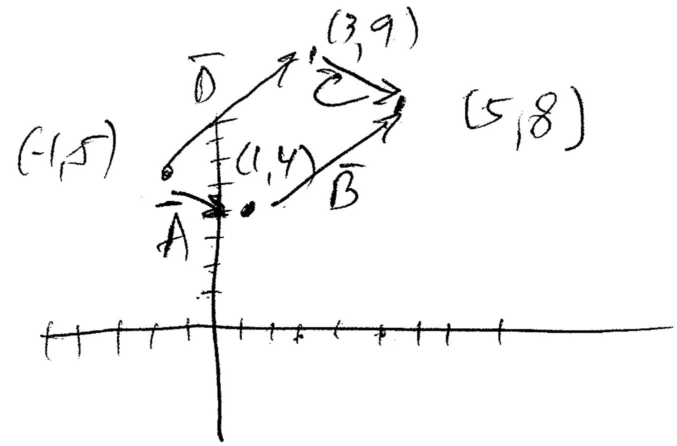
$$= a(15-12) - b(10-4) + c(6-3)$$

$$= 3a - 6b + 3c$$

If $a = b = c = 0$, then $\det A = 0$ & A ~~is~~ not invertible

If $a = b = 0$ & $c = 1$, then $\det A = 3$ & A is invertible.

3) (20pts) Consider the points in the plane: $(1,4),(5,8),(-1,5),(3,9)$.
a) These points determine a quadrilateral. Why is it a parallelogram?
b) Using determinants, find the area of this parallelogram.



$\bar{A} = (1,4) - (-1,5) = (2,-1)$
$\bar{C} = (5,8) - (3,9) = (2,-1) = \bar{A}$
$\bar{B} = (5,8) - (1,4) = (4,4)$
$\bar{D} = (3,9) - (-1,5) = (4,4) = \bar{B}$

$A = C$ & $B = D \Rightarrow$ the figure is a parallelogram

$$\text{Area} = \left|\det\left[\bar{A} \mid \bar{B}\right]\right| = \left|\det\begin{bmatrix} 2 & 4 \\ -1 & 4 \end{bmatrix}\right| = 12$$

Q: What do you call a message sent to you from two cities, at the same time?
A: A parallelogram.

4) (20pts) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ -1 & 4 & 5 & 1 \\ 2 & 1 & 2 & 1 \end{bmatrix}.$$

a) Find a basis for the column space of A.
b) What is the dimension of the column space of A?

Row reduce $A^T = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 4 & 1 \\ 3 & 5 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 6 & -3 \\ 0 & 8 & -4 \\ 0 & 2 & -1 \end{bmatrix} \begin{matrix} \\ II - 2I \\ III - 3I \\ IV - I \end{matrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix} \begin{matrix} \\ II/3 \\ III/4 \\ \\ \end{matrix}$

$\sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \} & v_1^T \\ & v_2^T \\ & \\ & \end{matrix}$

So $\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ & $\bar{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ are

a basis of the column space

AND it's dimension is 2.

5) (20pts) Consider the matrix

$$B = \begin{bmatrix} 4 & 3 & 3 & 4 \\ 1 & 1 & 3 & 3 \\ 3 & 2 & 0 & 1 \end{bmatrix}.$$

a) Find a basis of the null space of B.
b) What is the dimension of the null space of B?

Row reduce $B = \begin{bmatrix} 4 & 3 & 3 & 4 \\ 1 & 1 & 3 & 3 \\ 3 & 2 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 3 & 3 \\ 4 & 3 & 3 & 4 \\ 3 & 2 & 0 & 1 \end{bmatrix} \begin{matrix} II \\ I \\ \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & -1 & -9 & -8 \\ 0 & -1 & -9 & -8 \end{bmatrix} \begin{matrix} \\ II - 4I \\ III - 3I \end{matrix}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & 9 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ -II \\ III - II \end{matrix} \qquad \begin{matrix} x + y + 3z + 3w = 0 \\ y + 9z + 8w = 0 \end{matrix}$$

So use z, w as free variables.

Then $y = -9z - 8w$

$x = -y - 3z - 3w = 9z + 8w - 3z - 3w$

$= 6z + 5w$

Soln $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ to general vector in null space

is $\begin{bmatrix} 6z + 5w \\ -9z - 8w \\ z \\ w \end{bmatrix} = z \begin{bmatrix} 6 \\ -9 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 5 \\ -8 \\ 0 \\ 1 \end{bmatrix} = z\bar{v}_1 + w\bar{v}_2$

So $\{\bar{v}_1, \bar{v}_2\}$ is a basis & it is 2 dimensional.