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Math 225, Spring 2010
Midterm IIIA
April 15, 2010

NAME KEY

A) DO NOT OPEN YOUR EXAM UNTIL YOU ARE TOLD.

B) Be sure to write your name on this page.

C) This is a closed book, closed notes exam. No calculators allowed. Show all of your work or complete credit may not be given.

D) No cell phones allowed. Please turn them OFF during the exam.

E) No other electronic devices allowed e.g. MP3 players, PDAs, etc.

F) If you have a question, raise your hand and the proctor will come to you.

G) If you finish early, please hand in your exam and leave quietly in consideration of your fellow students. You will need to show a picture ID with a clear picture when you turn in your exam. When time is up, you will be instructed to put down your writing utensil and close your exam.

H) Every exam is worth a total of 100 points. Each exam has 6 pages including the cover sheet. Check to see that you have all the pages.

1) (20pts) Consider the 3×3 matrix

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 2 & 1 & 0 \\ -2 & 4 & -2 \end{bmatrix}.$$

Using the explicit values of the entries of the matrix, find the characteristic equation for this matrix.

$$0 = \begin{vmatrix} 2-\lambda & 3 & -1 \\ 2 & 1-\lambda & 0 \\ -2 & 4 & -2-\lambda \end{vmatrix} = (2-\lambda)\big((1-\lambda)(-2-\lambda)\big)$$
$$-3\,\big(2(-2-\lambda)\big)$$
$$-\big(8+2(1-\lambda)\big)$$

$\Longleftrightarrow$

$$0 = (2-\lambda)(\lambda^2+\lambda-2)+12+6\lambda$$
$$-8-2+2\lambda$$

$\Longleftrightarrow$

$$0 = -\lambda^3-\lambda^2+\underline{2\lambda}+2\lambda^2+\underline{2\lambda}-4$$
$$+4-2+\underline{8}\lambda$$

$\Longleftrightarrow$

$$0 = -\lambda^3+\lambda^2+12\lambda-2$$

Characteristic equation.

2) (20pts) Consider the 2×2 matrix $\begin{bmatrix} -2 & 2 \\ 4 & 1 \end{bmatrix}$.

a) Find the eigenvalues of this matrix and find an eigenvector for each of the eigenvalues.

b) Is there a basis of eigenvectors?

$$0 = \begin{vmatrix} -2-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = (-2-\lambda)(1-\lambda) - 8$$

$$= \lambda^2 + \lambda - 10$$

$$\lambda = \frac{-1 \pm \sqrt{1+40}}{2} = \frac{-1 \pm \sqrt{41}}{2}$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1+\sqrt{41}}{2}\begin{bmatrix} x \\ y \end{bmatrix} \iff -2x + 2y = \frac{-1+\sqrt{41}}{2} x$$

$$\iff \frac{-3-\sqrt{41}}{2} x + 2y = 0$$

$$\iff (-3-\sqrt{41})x + 4y = 0$$

eg $$\vec{v}_1 = \begin{bmatrix} \frac{-4}{-3-\sqrt{41}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3+\sqrt{41}} \\ 1 \end{bmatrix}$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \frac{-1-\sqrt{41}}{2}\begin{bmatrix} x \\ y \end{bmatrix} \iff -2x + 2y = \frac{-1-\sqrt{41}}{2} x$$

$$\iff (-3+\sqrt{41})x + 4y = 0$$

eg $$\vec{v}_2 = \begin{bmatrix} \frac{-4}{-3+\sqrt{41}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3-\sqrt{41}} \\ 1 \end{bmatrix}$$

3) (20pts) Consider the 2×2 matrix $B = \begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$. This matrix has distinct eigenvalues and there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of B.

a) Find the eigenvalues and at least one eigenvector for each of the eigenvalues of this matrix.

b) Using the eigenvectors in a), find a 2×2 matrix P such that $P^{-1}\, B\, P$ is a diagonal matrix.

$$(7-\lambda)(-8-\lambda) + 50 = 0$$

$$\lambda^2 + 8\lambda - 7\lambda - 56 + 50 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0 \qquad \lambda = 2, -3$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix} \iff 7x + 5y = 2x$$

$$\iff x = -y \text{ eg } \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \bar{v}_1$$

$$A\begin{bmatrix} x \\ y \end{bmatrix} = -3\begin{bmatrix} x \\ y \end{bmatrix} \iff 7x + 5y = -3x$$

$$\iff 10x + 5y = 0$$

$$\text{e.g. } y = 1,\ x = -\tfrac{1}{2}$$

$$\text{eg. } \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \bar{v}_2$$

$$P = [v_1 | v_2] = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

4) (20pts) Consider the matrix $B = \begin{bmatrix} 24 & -54 \\ 9 & -21 \end{bmatrix}$. This matrix has distinct eigenvalues and there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of B.

a) Find the two eigenspaces of B.

b) Find the cosine of the angle between the eigenspaces of B.

$$0 = (24 - \lambda)(-21 - \lambda) + 9 \cdot 54$$

$$= \lambda^2 - 3\lambda - 24 \cdot 21 + 9 \cdot 54$$

$$9 \cdot 54 = 486 \qquad 24 \cdot 21 = 24 + 480 = 504$$

$$0 = \lambda^2 - 3\lambda - 18 \iff \lambda = +6 \text{ or } \lambda = -3$$

$$= (\lambda - 6)(\lambda + 3)$$

$$B\begin{bmatrix} x \\ y \end{bmatrix} = 6\begin{bmatrix} x \\ y \end{bmatrix} \iff 24x - 54y = 6x$$

$$\iff 18x - 54y = 0 \qquad \bar{V}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

eg $y = 1$, $x = 3$

$$B\begin{bmatrix} x \\ y \end{bmatrix} = -3\begin{bmatrix} x \\ y \end{bmatrix} \iff 24x - 54y = -3x$$

$$\iff 27x - 54y = 0$$

eg $y = 1$, $x = 2$ $\qquad \bar{V}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\cos\theta = \frac{\bar{V}_1 \cdot \bar{V}_2}{\|\bar{V}_1\| \|\bar{V}_2\|} = \frac{6 + 1}{\sqrt{10}\sqrt{5}} = \frac{7}{\sqrt{50}}$$

5) (20pts) Consider the vectors $\mathbf{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Find a scalar c such that $\mathbf{v}$ and $\mathbf{w} - c\mathbf{v}$ are perpendicular to one another.

$$\bar{v} \perp \bar{w} - c\bar{v}$$

$$\Leftrightarrow \quad \bar{v} \cdot (\bar{w} - c\bar{v}) = 0$$

$$\Leftrightarrow \quad \bar{v} \cdot \bar{w} - c\,\bar{v} \cdot \bar{v} = 0$$

$$\Rightarrow \quad c = \frac{\bar{v} \cdot \bar{w}}{\bar{v} \cdot \bar{v}} = \frac{\bar{v} \cdot \bar{w}}{\|\bar{v}\|^2}$$

$$c = \frac{-6+4}{(\sqrt{10})^2} = \frac{-2}{10}$$