Solution to practice problem **4**.

Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

a. Find the characteristic polynomial and the eigenvalues of A.

Sol The characteristic polynomial of A is the determinant of $\begin{bmatrix} 1-\lambda & -1 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{bmatrix} =$ $(1-\lambda)^2(2-\lambda)$. So we have two distinct eigenvalues, say 1 and 2. Note that 1 has the algebraic multiplicity 2.

b. Find the eigenspace corresponding to each eigenvalue of A. Express the eigenspaces as the span of a set of vectors.

Sol First, let's try to find the eigenspace corresponding to $\lambda = 1$, which is by definition $\text{Nul}\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Since $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, it follows that

$$\text{Nul}\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Now let's consider the eigenspace corresponding to $\lambda = 2$, which is $\text{Nul}\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

Since $\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$, we get

$$\text{Nul}\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

c. For each eigenvalue of A, determine both algebraic and geometric multiplicities. Determine if A is diagonalizable.

Sol The algebraic multiplicity of $\lambda = 1$ is 2 (see part a above) and the geometric multiplicity of $\lambda = 1$ is 1 (see part b). The algebraic multiplicity of $\lambda = 2$ is 1 (see part a) and the geometric multiplicity of $\lambda = 2$ is 1 (see part b). Since the algebraic multiplicity of $\lambda = 1$ is **NOT** equal to the geometric multiplicity of $\lambda = 1$, we conclude that the matrix is **NOT** diagonalizable.