Solutions to Exam 2

Instructions: No calculators permitted on exam. Read each question carefully. To get full credit for an answer, you must provide full justification for the answer. Good luck!

1. (3 points each, 15 points total) True or False - no justifications required:

   (a) For any square matrix A and scalar k, $\det(kA)$ equals $k \det A$.

   False

   (b) For any $n \times n$ square matrices A and B, $\det(AB)$ equals $\det(B^T A)$.

   True

   (c) If $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p\}$ is a subset of a vector space V, then S forms a basis for Span$S$.

   False

   (d) In $\mathbb{R}^5$, there is a linearly independent set having 6 different 5-vectors.

   False

   (e) If A is an $n \times n$ square matrix with rank $A < n$, then the columns of A are linearly dependent.

   True

2. (20 points) Let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 5 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & -1 & -2 & \frac{53}{100} & \sqrt{2} \\ 0 & 0 & 3 & 1 & \frac{2}{7} \\ 0 & 0 & 0 & 4 & 1 \\ 2 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

(a) (8 points) Calculate the determinant of A.

Solution: Using the first row, $\det A = 3\begin{vmatrix} 0 & -1 \\ 5 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 5 \end{vmatrix} = 10.$

(b) (8 points) Calculate the determinant of B. *Hint:* B is *almost* upper triangular.

Solution: $R_5 \mapsto R_5 - 2R_1$ gives $\begin{bmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & -1 & -2 & \frac{53}{100} & \sqrt{2} \\ 0 & 0 & 3 & 1 & \frac{2}{7} \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & -8 & -6 \end{bmatrix}$ and $R_5 \mapsto R_5 + 2R_4$ gives $\begin{bmatrix} 1 & 0 & 0 & 4 & 5 \\ 0 & -1 & -2 & \frac{53}{100} & \sqrt{2} \\ 0 & 0 & 3 & 1 & \frac{2}{7} \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$. Since the determinant remains unchanged under replacement operations and the determinant of an upper triangular matrix is the product of diagonal entries, we get $\det B = 1 \cdot (-1) \cdot 3 \cdot 4 \cdot (-4) = 48$.

(c) (4 points) Calculate $\dim \operatorname{Col} B$, the dimension of the column space of B.

Solution: Since $\det B$ is nonzero, B is invertible. Therefore B has 5 pivot columns, which means that $\dim \operatorname{Col} B = 5$.

3. (15 points) Let $S = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \cdots, \boldsymbol{v}_n\}$ be a basis for a vector space V. Show that $T = \{\boldsymbol{v}_1, \boldsymbol{v}_1 + \boldsymbol{v}_2, \boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3, \cdots, \boldsymbol{v}_1 + \boldsymbol{v}_2 + \cdots + \boldsymbol{v}_n\}$ is also a basis for V.

Solution: First note that $\dim V = n$, because S contains n vectors. Since T contains n vectors as well, by the Basis Theorem, it suffices to show that T is linearly independent. Suppose that $c_1\boldsymbol{v}_1 + c_2(\boldsymbol{v}_1 + \boldsymbol{v}_2) + c_3(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3) + \cdots + c_n(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \cdots + \boldsymbol{v}_n) = \boldsymbol{0}$. Need to show that $c_1 = c_2 = c_3 = \cdots = c_n = 0$. Indeed,

$$\begin{aligned} & c_1\boldsymbol{v}_1 + c_2(\boldsymbol{v}_1 + \boldsymbol{v}_2) + c_3(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3) + \cdots + c_n(\boldsymbol{v}_1 + \boldsymbol{v}_2 + \cdots + \boldsymbol{v}_n) \\ = \; & (c_1 + \cdots + c_n)\boldsymbol{v}_1 + (c_2 + \cdots + c_n)\boldsymbol{v}_2 + (c_3 + \cdots + c_n)\boldsymbol{v}_3 + \cdots + c_n\boldsymbol{v}_n \end{aligned}$$

and S being linearly independent forces us to have

$$c_1 + \cdots + c_n = 0, \quad c_2 + \cdots + c_n = 0, \quad c_3 + \cdots + c_n = 0, \quad \cdots \quad , \quad c_n = 0.$$

In other words, $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ must be a solution of the homogeneous system $A\vec{x} = \vec{0}$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

is an $n \times n$ upper triangular matrix. Note that $\det A = 1$, so A is invertible, and hence $\vec{0}$ is the only solution to $A\vec{x} = \vec{0}$. This shows that T is linearly independent.

4. (10 points each, 20 points total)

(a) Let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \leq 1 \right\}$, the set of points inside and on the unit circle in the xy-plane. Determine whether H is a subspace of $\mathbb{R}^2$.

Solution: Consider $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then their sum $\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is outside the unit circle. So H is <u>not</u> a subspace of $\mathbb{R}^2$. For another example, consider $c = 100$ and $c\vec{u} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$. It is clear that $c\vec{u}$ is not inside the circle.

(b) Show that the set K of all polynomials of the form $p(t) = a + bt^2 + ct^4$, where $a, b, c \in \mathbb{R}$, is a subspace of $\mathbb{P}_4$.

Solution: Clearly the zero polynomial $z(t) \equiv 0$ is in K, since $z(t) = 0+0\cdot t^2+0\cdot t^4$. If $p(t) = a + bt^2 + ct^4$ and $q(t) = a' + b't^2 + c't^4$, then $p(t) + q(t) = (a + a') + (b + b')t^2 + (c + c')t^4$ is also in K. Finally, for any constant k, $kp(t) = ka + kbt^2 + kct^4$ is again in K.

Another way: note that $K = \text{Span}\{\alpha(t), \beta(t), \gamma(t)\}$, where $\alpha(t) = 1$, $\beta(t) = t$, and $\gamma(t) = t^2$. In other words, K is the subspace of $\mathbb{P}_4$ spanned by $\alpha(t)$, $\beta(t)$, and $\gamma(t)$.

5. (15 points) Let $A = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix}$.

(a) (10 points) Find a basis for Col A.

Solution: It is not difficult to see that A becomes $\begin{bmatrix} 1 & 0 & \frac{4}{5} & 2 \\ 0 & 1 & \frac{1}{5} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ after suitable row operations. This tells us that Column 1 and Column 2 are pivotal and hence $\left\{ \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \right\}$ can be a basis for Col A.

(b) (5 points) What is the dimension of Nul A?

Solution: By the rank theorem, the nullity of $A = 4 - \dim \operatorname{Col} A = 4 - 2 = 2$.

6. (15 points) Consider a 1×3 matrix $A = \begin{bmatrix} 1 & -3 & -2 \end{bmatrix}$. Describe $\operatorname{Nul} A$ as the span of some vectors.

Solution: Let's first describe the solution set of the system $A\vec{x} = \vec{0}$. Since only Column 1 is pivotal, x_3 and x_2 are free, while $x_1 = 3x_2 + 2x_3$. So the general solution can be expressed as $\begin{bmatrix} 3x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ with x_2 and x_3 being free.

This simply means that $\operatorname{Nul} A = \operatorname{Span} \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.