SOLUTIONS TO SELECTED PROBLEMS IN SECTION 1.2

1. **a**. in reduced echelon form.
b. in reduced echelon form.
c. not even in echelon form: R_3 is a row of zeros and it is above R_4 which is a nonzero row. If you interchange R_3 and R_4, then it becomes a reduced echelon matrix.
d. in echelon form but not in reduced echelon form. For example, the leading entry in R_2 is 2, which is not 1.

2. **a**. in reduced echelon form.
b. in echelon form but not in reduced echelon form: (2,2) entry is the leading entry in R_2 but it is not the only nonzero in its column, say column 2.
c. not even in echelon form: you can find a 'bad' number in column 1.
d. in echelon form but not in reduced echelon form. Having a zero column as the first column doesn't matter.

4. First of all,

$$\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix} \xrightarrow[R_3 \mapsto R_3 - 5R_1]{R_2 \mapsto R_2 - 3R_1} \begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{bmatrix}.$$

$R_2 \mapsto -\frac{1}{4}R_2$ gives $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{bmatrix}$ and the replacement operation $R_3 \mapsto R_3 + 8R_2$ in turn gives $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix}$, which becomes $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix}$ after $R_1 \mapsto R_1 - 3R_3$.
Now

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{bmatrix} \xrightarrow{R_3 \mapsto -\frac{1}{10}R_3} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2 \mapsto R_2 - 3R_3]{R_1 \mapsto R_1 + 2R_3} \begin{bmatrix} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}.$$

Pivot columns are C_1, C_2, and C_4.

9. We get

$$\begin{bmatrix} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 7 & -6 \\ 0 & 1 & -6 & 5 \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 + 2R_2} \begin{bmatrix} 1 & 0 & -5 & 4 \\ 0 & 1 & -6 & 5 \end{bmatrix},$$

so x_1, x_2 are basic and x_3 is free. The general solution is given by

$$\begin{cases} x_1 &= 4 + 5x_3 \\ x_2 &= 5 + 6x_3 \\ x_3 & \text{is free} \end{cases}.$$

14. The reduced echelon form of the given matrix is obtained as following:

$$\begin{bmatrix} 1 & 2 & -5 & -6 & 0 & -5 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 7 & 0 & 0 & -9 \\ 0 & 1 & -6 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and hence x_1, x_2, x_5 are basic variables, while x_3, x_4 are free. The general solution is

$$\begin{cases} x_1 &= -9 - 7x_3 \\ x_2 &= 2 + 6x_3 + 3x_4 \\ x_3 & \text{is free} \\ x_4 & \text{is free} \\ x_5 &= 0 \end{cases}.$$

17. First find the reduced echelon form of the given matrix:

$$\begin{bmatrix} 2 & 3 & h \\ 4 & 6 & 7 \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1} \begin{bmatrix} 2 & 3 & h \\ 0 & 0 & 7 - 2h \end{bmatrix} \xrightarrow{R_1 \mapsto \frac{1}{2}R_1} \begin{bmatrix} 1 & \frac{3}{2} & \frac{h}{2} \\ 0 & 0 & 7 - 2h \end{bmatrix}.$$

This tells us that the original linear system is equivalent to the modified system

$$\begin{array}{rcl} x_1 + \frac{3}{2}x_2 &=& \frac{h}{2} \\ 0 &=& 7 - 2h \end{array},$$

which has a solution (actually infinitely many solutions) if and only if $h = \frac{7}{2}$. In other words, this system is consistent if and only if $h = \frac{7}{2}$.

18. Given matrix becomes

$$\begin{bmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - 5R_1} \begin{bmatrix} 1 & -3 & -2 \\ 0 & h + 15 & 3 \end{bmatrix}$$

and hence the original system is equivalent to

$$\begin{array}{rcl} x_1 \quad -3x_2 &=& -2 \\ (h + 15)x_2 &=& 3 \end{array}.$$

Look at the second equation. If $h = -15$ so that the coefficient of x_2 is 0, then there is no x_2 that makes the equation hold (0 cannot be equal to 3). On the other hand, if $h \neq -15$, then we can perform one more row operation to get

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & h + 15 & 3 \end{bmatrix} \xrightarrow{R_2 \mapsto \frac{1}{h+15}R_2} \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & \frac{3}{h+15} \end{bmatrix},$$

which means the solution of the system is given by $x_1 = -2$, $x_2 = \frac{3}{h+15}$. So conclusion: this system is consistent if and only if $h \neq -15$.

Another solution: Given matrix becomes

$$\begin{bmatrix} 1 & -3 & -2 \\ 5 & h & -7 \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - 5R_1} \begin{bmatrix} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{bmatrix}.$$

$\boxed{\text{Case 1}}$ If $h + 15 = 0$, then

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{bmatrix} \xrightarrow{R_2 \mapsto \frac{1}{3}R_2} \begin{bmatrix} 1 & -3 & -2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 + 2R_2} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the rightmost column is a pivot column. In short, if $h = -15$, then the system is inconsistent.

$\boxed{\text{Case 2}}$ If $h + 15 \neq 0$, then we can divide R_2 by $h + 15$ and hence

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & h+15 & 3 \end{bmatrix} \xrightarrow{R_2 \mapsto \frac{1}{h+15}R_2} \begin{bmatrix} 1 & -3 & -2 \\ 0 & 1 & \frac{3}{h+15} \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 + 3R_2} \begin{bmatrix} 1 & 0 & -2 + \frac{9}{h+15} \\ 0 & 1 & \frac{3}{h+15} \end{bmatrix}.$$

The rightmost column of the last matrix is not a pivot column. Therefore the system is consistent. In short, if $h \neq -15$, then the system is consistent.

So conclusion: this system is consistent if and only if $h \neq -15$.

21. **a**. False. Refer to Theorem 1.
b. False. You can apply it to coefficient matrix, too.
c. True. A free variable corresponds to a nonpivot column
d. True. Actually, this is just a matter definition.
e. False. Consider, for example, the following matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 5 & 0 \end{bmatrix}$$

which can be row reduced into

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

using $R_2 \mapsto \frac{1}{5}R_2$. This system is consistent, because the rightmost column is not a pivot column. In fact, this system has infinitely many solutions, because it has two free variables, say x_2 and x_3. The variables x_1 and x_4 are basic.