SOLUTIONS TO SELECTED PROBLEMS IN SECTION 1.3

1. First note that the text book uses **bold** letters to denote vectors. We keep using letters with an arrow as we did in class.
$\vec{u}+\vec{v}=\begin{bmatrix} -4 \\ 1 \end{bmatrix}$, $-2\vec{v}=(-2)\vec{v}=\begin{bmatrix} 6 \\ 2 \end{bmatrix}$, $\vec{u}-2\vec{v}=\vec{u}+(-2)\vec{v}=\begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

5. The vector equation is equivalent to the system

$$\begin{array}{rrcr} 6x_1 & -3x_2 & = & 1 \\ -x_1 & +4x_2 & = & -7 \\ 5x_1 & & = & -5 \end{array},$$

which in turn equivalent to the following matrix-vector representation:

$$\begin{bmatrix} 6 & -3 \\ -1 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}.$$

8. $\vec{w}$ represents the fourth vertex of the parallelogram having $2\vec{v}$ and $-\vec{u}$ as its two incident sides. So $\vec{w}=2\vec{v}-\vec{u}$ or $-\vec{u}+2\vec{v}$. Similarly, $\vec{x}=2\vec{v}-2\vec{u}$. It appears $\vec{y}$ is placed at the fourth vertex of the parallelogram having $-2\vec{u}$ and $3.5\vec{v}$ as its sides, so $\vec{y}=3.5\vec{v}-2\vec{u}$. Use a little bit of imagination (neither $-3\vec{u}$ nor $4\vec{v}$ is shown) to get $\vec{z}=-3\vec{u}+4\vec{v}$.

11. Check whether the system associated with the augmented matrix

$$\left[\begin{array}{rrr:r} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right]$$

is consistent or not. The matrix becomes

$$\left[\begin{array}{rrr:r} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \xrightarrow{R_2 \mapsto R_2+2R_1} \left[\begin{array}{rrr:r} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{array}\right] \xrightarrow{R_3 \mapsto R_3-2R_2} \left[\begin{array}{rrr:r} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and the rightmost column is not pivotal. Thus the system is consistent and hence $\vec{b}$ is a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$. In other words, $\vec{b} \in \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$. Coefficients? We have more than one triple of coefficients which expresses $\vec{b}$ in terms of $\vec{a}_1, \vec{a}_2, \vec{a}_3$. For example, $\vec{b}=2\vec{a}_1+3\vec{a}_2+0\vec{a}_3=2\vec{a}_2+3\vec{a}_2$. $\vec{b}=-3\vec{a}_1-\vec{a}_2+\vec{a}_3$ also works. Can you find all such triples? I will give the answer, you justify why:

$$\vec{b}=(2-5t)\vec{a}_1+(3-4t)\vec{a}_2+t\vec{a}_3.$$

Note that two previous triples correspond to $t = 0, 1$, respectively.

21. It suffices to show that the system associated with the following augmented matrix

$$\begin{bmatrix} 2 & 2 & \vdots & h \\ -1 & 1 & \vdots & k \end{bmatrix}$$

is consistent no matter what values we choose for h and k. After some proper row operations, the matrix becomes

$$\begin{bmatrix} 2 & 2 & \vdots & h \\ -1 & 1 & \vdots & k \end{bmatrix} \xrightarrow[\text{row operations}]{\text{many}} \begin{bmatrix} 1 & 0 & \vdots & \frac{h-2k}{4} \\ 0 & 1 & \vdots & \frac{h+2k}{4} \end{bmatrix}$$

and the rightmost column of the last matrix is not pivotal for any choice of h, k (We have two pivot columns, say column 1 and 2). This observation completes the proof.

24. **a**. True.
b. True. One can prove this using properties in p.32.
c. False. They all could be zero. That's why we have $\vec{0}$ in the span of any collection of vectors.
d. True. If both $\vec{u}$ and $\vec{v}$ are zero vectors, the statement is no longer true.
e. True. The system is consistent if and only if $\vec{b} \in \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

29. Total mass = 2+5+2+1=10.

$$\begin{aligned} \text{Center of gravity } \vec{v} &= \frac{1}{10}\left(2\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} + 2\begin{bmatrix} -4 \\ -3 \\ -1 \end{bmatrix} + \begin{bmatrix} -9 \\ 8 \\ 6 \end{bmatrix}\right) \\ &= \frac{1}{10}\left(\begin{bmatrix} 10 \\ -8 \\ 6 \end{bmatrix} + \begin{bmatrix} 20 \\ 15 \\ -10 \end{bmatrix} + \begin{bmatrix} -8 \\ -6 \\ -2 \end{bmatrix} + \begin{bmatrix} -9 \\ 8 \\ 6 \end{bmatrix}\right) \\ &= \frac{1}{10}\begin{bmatrix} 13 \\ 9 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1.3 \\ .9 \\ 0 \end{bmatrix}. \end{aligned}$$