SOLUTIONS TO SELECTED PROBLEMS IN SECTION 1.4

1. undefined.

2. undefined.

4. The product would be a 2×1 matrix, and it is

$$\begin{bmatrix} 8 \cdot 1 + 3 \cdot 1 + (-4) \cdot 1 \\ 5 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$

13. Yes. The augmented matrix associated with this problem is

$$\begin{bmatrix} 3 & -5 & \vdots & 0 \\ -2 & 6 & \vdots & 4 \\ 1 & 1 & \vdots & 4 \end{bmatrix}$$

which becomes

$$\begin{bmatrix} 1 & 0 & \vdots & \frac{5}{2} \\ 0 & 1 & \vdots & \frac{3}{2} \\ 0 & 0 & \vdots & 0 \end{bmatrix}$$

after some row operations. Since the rightmost column of the last matrix is not a pivot column, the system is consistent and hence $\vec{u}$ in the span of columns of A.

14. No. $\vec{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} \right\}$ if and only if there are scalars c_1, c_2, c_3 such that

$$\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix},$$

which is equivalent to saying that the system

$$\begin{array}{rrrcr} 5c_1 & +8c_2 & +7c_3 & = & 2 \\ & c_2 & -c_3 & = & -3 \\ c_1 & +3c_2 & & = & 2 \end{array}$$

is consistent. Now the augmented matrix

$$\begin{bmatrix} 5 & 8 & 7 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & -3 \\ 1 & 3 & 0 & \vdots & 2 \end{bmatrix}$$

associated with the above system becomes

$$\begin{bmatrix} 1 & 0 & 3 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$$

and the rightmost column is pivotal. This tells us that the system is inconsistent and hence $\vec{u}$ is not in the span of column vectors of A.

Remark In general, if $\vec{u}, \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ are n-vectors (having n entries), how can we check whether $\vec{u}$ is in $\text{Span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ or not? There is a shortcut (Imitate the solution above to see why it works): form an $n \times (k+1)$ matrix A whose columns consist of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ and $\vec{u}$. That is,

$$A = \begin{bmatrix} | & | & & | & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k & \vec{u} \\ | & | & & | & | \end{bmatrix}$$

(Caution: $\vec{u}$ must be the rightmost column in A!). Obtain the reduced row echelon form of A. If the rightmost column of the reduced echelon matrix is pivotal, then $\vec{u} \notin \text{Span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$. If the rightmost column of the reduced echelon matrix is not pivotal, then $\vec{u} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$.

25. The equation

$$\begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$$

holds for c_1, c_2, c_3 if and only if equations

$$\begin{array}{rrrcr} 4c_1 & -3c_2 & +c_3 & = & -7 \\ 5c_1 & -2c_2 & +5c_3 & = & -3 \\ -6c_1 & +2c_2 & -3c_3 & = & 10 \end{array} \qquad (*)$$

hold for c_1, c_2, c_3. Given equality

$$\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$$

tells us that three equations in $(*)$ are all satisfied when we plug in $(-3, -1, 2)$, so we can take $(c_1, c_2, c_3) = (-3, -1, 2)$.

Remark: this does NOT tell us whether $(c_1, c_2, c_3) = (-3, -1, 2)$ is the only such triple, though it is. This question is about uniqueness of a solution of given system and we will develop some methods in chapter 2 to answer that kind of question.