SOLUTIONS TO SELECTED PROBLEMS IN SECTION 1.5

1. Since any homogeneous system has at least one solution (trivial solution), the given system has a nontrivial solution if and only if it has infinitely many solutions (Remember, the number of solutions of a linear system must be 0,1, or ∞). In other words, given homogeneous system has a nontrivial solution if and only if it has at least one free variable. Now the augmented matrix associated with the system is

$$\begin{bmatrix} 2 & -5 & 8 & \vdots & 0 \\ -2 & -7 & 1 & \vdots & 0 \\ 4 & 2 & 7 & \vdots & 0 \end{bmatrix}$$

which, after some proper row operations, becomes

$$\begin{bmatrix} 1 & 0 & \frac{17}{8} & \vdots & 0 \\ 0 & 1 & -\frac{3}{4} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

This reduced row echelon form tells us that x_3 is a free variable, while x_2, x_1 are basic.

6. Let's solve the system first. The augmented matrix associated with the system is

$$\begin{bmatrix} 1 & 3 & -5 & \vdots & 0 \\ 1 & 4 & -8 & \vdots & 0 \\ -3 & -7 & 9 & \vdots & 0 \end{bmatrix}$$

and it becomes

$$\begin{bmatrix} 1 & 0 & 4 & \vdots & 0 \\ 0 & 1 & -3 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

after some row operations. Therefore the solution is

$$\begin{cases} x_2 & = & -4x_3 \\ x_2 & = & 3x_3 \\ x_3 & & \text{is free} \end{cases}.$$

If we use t to denote the value of x_3, the general solution would be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4t \\ 3t \\ t \end{bmatrix} = t \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}.$$

The last expression is what the problem asked. Note that the solution set is Span $\left\{ \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} \right\}$.

8. What's given is just the coefficient matrix. The augmented matrix is

$$\begin{bmatrix} 1 & -2 & -9 & 5 & \vdots & 0 \\ 0 & 1 & 2 & -6 & \vdots & 0 \end{bmatrix}$$

and it is row-reduced to

$$\begin{bmatrix} 1 & 0 & -5 & -7 & \vdots & 0 \\ 0 & 1 & 2 & -6 & \vdots & 0 \end{bmatrix}.$$

Now we have two free variables (x_3, x_4) and two basic variables (x_1, x_2). If we put $x_4 = t$ and $x_3 = s$, then the general solution will look like

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5s + 7t \\ -2s + 6t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}.$$

The solution set can also be described by Span $\left\{ \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}$.

11. Note first that the matrix is *not* in a reduced echelon form (-2 and 3 are bad numbers). The augmented matrix

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

is transformed into

$$\begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 5 & \vdots & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Note that x_1, x_3, x_5 are basic and x_2, x_4, x_6 are free. If we put $x_2 = r, x_4 = s, x_6 = t$, then the general solution will be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4r - 5t \\ r \\ t \\ s \\ 4t \\ t \end{bmatrix} = r \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$$

The solution set is Span $\left\{ \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix} \right\}$.

26. We need to show the following two statements:

I. If $A\vec{x} = \vec{b}$ has a unique solution, then $A\vec{x} = \vec{0}$ has only the trivial solution.
II. If $A\vec{x} = \vec{0}$ has only the trivial solution, then $A\vec{x} = \vec{b}$ has a unique solution.

In the problem, the existence of a solution of $A\vec{x} = \vec{b}$ is assumed. Let $\vec{p}$ be a solution of the system, that is, $A\vec{p} = \vec{b}$.

Proof of I. Suppose $\vec{p}$ is the only solution of $A\vec{x} = \vec{b}$. If there were a vector $\vec{w}$, not equal to $\vec{0}$, such that $A\vec{w} = \vec{0}$, then $\vec{p} + \vec{w}$ must be another solution of $A\vec{x} = \vec{b}$. Since we assumed that $\vec{p}$ is the only solution of $A\vec{x} = \vec{b}$, this would mean that $\vec{p} + \vec{w} = \vec{p}$, or $\vec{w} = \vec{0}$. This is a contradiction, because $\vec{w}$ cannot be both a nonzero vector and a zero vector at the same time.

Proof of II. Suppose the system $A\vec{x} = \vec{b}$ has a solution $\vec{q}$ which is different from $\vec{p}$. Since every solution $\vec{w}$ of the system $A\vec{x} = \vec{b}$ must be of the form $\vec{p} + \vec{v}_h$ with $\vec{v}_h$ being a solution of $A\vec{x} = \vec{0}$, our $\vec{q}$, as a solution of $A\vec{x} = \vec{b}$, must be written as $\vec{q} = \vec{p} + \vec{v}_h$, where $A\vec{v}_h = \vec{0}$. Since the homogeneous system $A\vec{x} = \vec{0}$ has only the trivial solution, $\vec{v}_h$ must be $\vec{0}$. In other words, $\vec{q} = \vec{p}$. Contradiction.

29. (a) No. If A has three pivot positions, we have no free variables. Therefore, the number of solutions of $A\vec{x} = \vec{0}$ is 1.

(b) Yes. The augmented matrix associated with the system has exactly three pivots: the augmented matrix is a 3×4 matrix and each row (and each column) has at most one pivot position. Since A, the coefficient matrix part of the augmented matrix, has already 3 pivot positions, this would mean the rightmost column of the augmented matrix cannot be a pivot column. Therefore, the system is consistent.

30. (a) Yes. Since A has only two pivot positions, the system $A\vec{x} = \vec{0}$ has one free variable and two basic variables. Therefore, $A\vec{x} = \vec{0}$ has infinitely many solutions.

(b) No. Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This example serves as a counter-example. Check it yourself.

31. (a) No. The equation $A\vec{x} = \vec{0}$ has no free variables.

(b) No. Consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Check why this gives you a counter-example.

32. (a) Yes. Since A has only two pivot positions, the system $A\vec{x} = \vec{0}$ has two free variables and two basic variables. Therefore, $A\vec{x} = \vec{0}$ has infinitely many solutions.

(b) Yes. The augmented matrix associated with the system $A\vec{x} = \vec{b}$ has two rows and hence the matrix can have at most 2 pivot positions. Since A has two pivot positions already, the rightmost column of the augmented matrix cannot be pivotal. Therefore the system is consistent for any choice of $\vec{b}$.

34. $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ works. In fact, the solution set is Span $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}$.

36. There are infinitely many solutions. In fact, any 3×3 matrix A with the following property is O.K.:

For each row of A, the dot product of the row and the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ equals 0.

For example, you can take $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & -2 & -8 \\ 0 & 1 & 2 \end{bmatrix}$. You can produce another example, of course. If you have one, find the reduced echelon form of the matrix. If it has 3 pivot columns, then it means something is wrong. Why?