## SOLUTIONS TO SELECTED PROBLEMS IN SECTION 1.7

**1**. Consider the matrix $A$ whose columns consist of the given vectors, say $A = \begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -8 \end{bmatrix}$.

The augmented matrix associated with the system $A\vec{x} = \vec{0}$ is therefore $\begin{bmatrix} 5 & 7 & 9 & \vdots & 0 \\ 0 & 2 & 4 & \vdots & 0 \\ 0 & -6 & -8 & \vdots & 0 \end{bmatrix}$, which becomes $\begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \end{bmatrix}$. The last matrix has no free variables and that means the trivial solution is the only solution the system has. Therefore $S = \left\{ \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix} \right\}$ is linearly independent.

**4**. The set $S = \left\{ \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -8 \end{bmatrix} \right\}$ is linearly independent. Let's count the number of free variables of the homogeneous system $A\vec{x} = \vec{0}$, where $A = \begin{bmatrix} -1 & -2 \\ 4 & -8 \end{bmatrix}$. The augmented matrix is nothing but $\begin{bmatrix} -1 & -2 & \vdots & 0 \\ 4 & -8 & \vdots & 0 \end{bmatrix}$ and it is easy to see that the last matrix row-reduces to $\begin{bmatrix} 1 & 0 & \vdots & 0 \\ 0 & 1 & \vdots & 0 \end{bmatrix}$. This shows that the system has no free variables.

**6**. Let $A = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$. The columns are linearly independent if and only if the homogeneous system $A\vec{x} = \vec{0}$ has no free variables. Now the augmented matrix associated with the system $A\vec{x} = \vec{0}$ is $\begin{bmatrix} -4 & -3 & 0 & \vdots & 0 \\ 0 & -1 & 4 & \vdots & 0 \\ 1 & 0 & 3 & \vdots & 0 \\ 5 & 4 & 6 & \vdots & 0 \end{bmatrix}$, which reduces to $\begin{bmatrix} 1 & 0 & 0 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$. Thus the system has no free variables and hence the set of columns of $A$ is linearly independent.

**7**. Linearly dependent. You have four 3-vectors. See Theorem 8 in p.69. Problem **8** was not a homework problem, but exactly the same reasoning as in problem **7** applies here.

**10**. (a) The following statements are equivalent, that is, they are either all true or all false:

(1) $\vec{v}_3 \in \text{Span}\{\vec{v}_1, \vec{v}_2\}$,
(2) there are numbers $x_1, x_2$ such that $x_1\vec{v}_1 + x_2\vec{v}_2 = \vec{v}_3$,
(3) the system $\begin{bmatrix} 1 & -2 & \vdots & 2 \\ -5 & 10 & \vdots & -9 \\ -3 & 6 & \vdots & h \end{bmatrix}$ is consistent,
(4) the rightmost column of $\begin{bmatrix} 1 & -2 & \vdots & 2 \\ -5 & 10 & \vdots & -9 \\ -3 & 6 & \vdots & h \end{bmatrix}$ is not a pivot column.

So let's find the reduced echelon form of $\begin{bmatrix} 1 & -2 & \vdots & 2 \\ -5 & 10 & \vdots & -9 \\ -3 & 6 & \vdots & h \end{bmatrix}$. If we use row operations $R_2 \mapsto R_2 + 5R_1$ and $R_3 \mapsto R_3 + 3R_1$, then we get $\begin{bmatrix} 1 & -2 & \vdots & 2 \\ 0 & 0 & \vdots & 1 \\ 0 & 0 & \vdots & h+6 \end{bmatrix}$ as a result. Now, row operations $R_1 \mapsto R_1 - 2R_2$ and $R_3 \mapsto R_3 - (h+6)R_2$ transform $\begin{bmatrix} 1 & -2 & \vdots & 2 \\ 0 & 0 & \vdots & 1 \\ 0 & 0 & \vdots & h+6 \end{bmatrix}$ into $\begin{bmatrix} 1 & -2 & \vdots & 0 \\ 0 & 0 & \vdots & 1 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$, showing the rightmost column is a pivot column. Therefore we have somewhat perplexing conclusion: no matter what $h$ is, $\vec{v}_3$ is not in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$.

(b) Let $A = \begin{bmatrix} 1 & -2 & 2 \\ -5 & 10 & -9 \\ -3 & 6 & h \end{bmatrix}$. The following statements are equivalent:

(1) $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent,
(2) the system $A\vec{x} = \vec{0}$ has a nontrivial solution,
(3) the system $A\vec{x} = \vec{0}$ has infinitely many solutions,
(4) the system $A\vec{x} = \vec{0}$ has at least one free variable.

To see the number of free variables, let's find the reduced echelon form of $A$ (Note: to count the number of free variables, just the *coefficient* matrix is enough. You don't have to consider the augmented one). Actually, this job was done in part (a) - just remove the vertical dots in the augmented matrix that appeared in (a) - and we see that $x_2$ is a free variable. So again a little bit perplexing conclusion: $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent no matter what values $h$ has. Note also that, for any $h$, $2 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 = \vec{0}$, showing $0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + 0 \cdot \vec{v}_3 = \vec{0}$ is **NOT** the only way to make $x_1 \cdot \vec{v}_1 + x_2 \cdot \vec{v}_2 + x_3 \cdot \vec{v}_3 = \vec{0}$.

**12**. We have to find the value(s) of $h$ for which the homogeneous system $\begin{bmatrix} 2 & -6 & 8 & \vdots & 0 \\ -4 & 7 & h & \vdots & 0 \\ 1 & -3 & 4 & \vdots & 0 \end{bmatrix}$ has at least one free variable. It is not difficult to see that it has two pivot positions and $x_3$ is a pivot column no matter what $h$ is. Therefore we conclude that given vectors are linearly dependent for any choice of $h$.

**21**. **a**. False. The columns of a matrix $A$ are linearly independent if the equation $A\vec{x} = \vec{0}$ has *only* the trivial solution.
**b**. False. Let's consider $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. I will leave it for you to check $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, but $\vec{v}_3$ cannot be expressed as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.
**c**. True. See Theorem 8 in p.69.
**d**. True. Since $\{\vec{x}, \vec{y}, \vec{z}\}$ is linearly dependent, there are numbers $c_1, c_2, c_3$, not all zero, such that $c_1\vec{x} + c_2\vec{y} + c_3\vec{z} = \vec{0}$. Here you might claim that we are done because we have $\vec{z} = -\frac{c_1}{c_3}\vec{x} - \frac{c_2}{c_3}\vec{y}$. Not really, because this argument would be absurd if $c_3$ happens to be 0. This idea, however, is still valuable, because if you can somehow show that $c_3$ is not zero, then the argument is perfect. So now let's show that $c_3$ can't be zero. Suppose, for contradiction, that $c_3$ equals 0, then it would mean that $c_1\vec{x} + c_2\vec{y} = \vec{0}$. Since $c_1 = c_2 = 0$ is the only possible way to have $c_1\vec{x} + c_2\vec{y} = \vec{0}$ (because $\{\vec{x}, \vec{y}\}$ is linearly independent), we therefore have $c_1 = c_2 = c_3 = 0$. It is a contradiction because at least one of $\{c_1, c_2, c_3\}$ must be nonzero. This contradiction came from our assumption that $c_3 = 0$, so $c_3$ cannot be zero.

**27**. All five columns. Let $A$ be the $7 \times 5$ matrix mentioned in the problem. The columns of $A$ are linearly independent if and only if the system $A\vec{x} = \vec{0}$ has no free variables, that is, all columns of the coefficient matrix $A$ must be pivotal.

**33**. True. You can express the zero vector as $\vec{0} = 2\vec{v}_1 + \vec{v}_2 - \vec{v}_3 + 0\vec{v}_4$. This shows that $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 + 0\vec{v}_4$ is **NOT** the only way to get $\vec{0}$.

**34**. True. Any set of vectors containing $\vec{0}$ is linearly dependent.

**35**. False. Consider $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

**36**. False. Consider $\vec{v}_1 = \vec{v}_2 = \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.