SOLUTIONS TO SELECTED PROBLEMS IN SECTION 4.1

1. **a**. Yes. Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} z \\ w \end{bmatrix}$, with $x, y, z, w \geq 0$. Then $\vec{u} + \vec{v} = \begin{bmatrix} x+z \\ y+w \end{bmatrix}$ and $x + y \geq 0, z + w \geq 0$, being sums of nonnegative numbers. This shows $\vec{u} + \vec{v}$ is in V.
b. Take $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $c = -2$, for example. Then $c\vec{u} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$, which is not in V.

2. **a**. Yes. Let $\vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ with $xy \geq 0$. For any $c \in \mathbb{R}$, $c\vec{u} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ and the product of two components of $c\vec{u}$ is $cx \cdot cy = c^2xy \geq 0$, because $c^2 \geq 0$ and $xy \geq 0$. This tells us that $c\vec{u}$ is in W, since W consists of all 2-vectors $\vec{u}$ such that the product of two components of $\vec{u}$ is nonnegative.
b. Consider $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$, then $\vec{u} + \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which is not in W.

3. Consider $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then their sum $\vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is outside the unit circle. For another example, consider $c = 100$ and $c\vec{u} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$. It is clear that $c\vec{u}$ is not inside the circle.

6. $z(t) = 0$, the zero polynomial, is not of the form $p(t) = a + t^2$, so given collection is not a subspace of $\mathbb{P}_n$.

8. Let $H = \{p(t) : p(0) = 0, \quad p(t) : \text{polynomial}\}$. We claim that H is a subspace. First of all, clearly the zero polynomial is in H: the zero polynomial $z(t) = 0$ gives 0 no matter what t is. In particular, $z(0) = 0$. Second, suppose we have two polynomials $p(t)$ and $q(t)$ in H, so $p(0) = q(0) = 0$. Their sum $p(t) + q(t)$ equals 0 when we put $t = 0$, because $p(0) + q(0) = 0 + 0 = 0$. Finally, for any scalar $c \in \mathbb{R}$ and $p(t) \in H$, $cp(0) = c \cdot 0 = 0$, which means $cp(t)$ is in H.

9. $H = \text{Span}\left\{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right\}$, is a subspace of $\mathbb{R}^3$ because the span of any set of vectors is always a subspace (Theorem 1, p.221).

10. $H = \text{Span}\left\{\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}\right\}$. Again we may apply Theorem 1, p.221, to show that H is a subspace of $\mathbb{R}^3$.

11. $H = \text{Span}\left\{\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\}$. The same reason as above.

12. $H = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}\right\}$. Still the same argument works. Note that this time H is a subspace of $\mathbb{R}^4$.

16. Probably the easiest way to show that this set is not a subspace of $\mathbb{R}^3$ is by showing $\vec{0}$ is not in W. If $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ were in W, it must be of the form $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$, forcing us to have $-a+1 = 0$, $a - 6b = 0$, and $2b + a = 0$. First two equations tell us $a = 1$ and hence $b = \frac{1}{6}$. But the last means $b = -\frac{1}{2}$. This contradiction is due to our assumption that $\vec{0}$ is of the form $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$. In other words, $\vec{0}$ cannot be in W.

18. $W = \text{Span}\left\{\begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right\}$.