SOLUTIONS TO SELECTED PROBLEMS IN SECTION 4.3

1. Suppose $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The sum on the left is $\begin{bmatrix} c_1 + c_2 + c_3 \\ c_2 + c_3 \\ c_3 \end{bmatrix}$ and the equality forces all c_1, c_2, c_3 to be zero. This shows that the set is linearly independent. Now for arbitrary $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$, the augmented matrix $\begin{bmatrix} 1 & 1 & 1 & \vdots & x \\ 0 & 1 & 1 & \vdots & y \\ 0 & 0 & 1 & \vdots & z \end{bmatrix}$ row-reduces to $\begin{bmatrix} 1 & 0 & 0 & \vdots & x-y \\ 0 & 1 & 0 & \vdots & y-z \\ 0 & 0 & 1 & \vdots & z \end{bmatrix}$ and the rightmost column of the last matrix is nonpivotal no matter what x, y, z are. Referring to problem **25 d** above, we conclude that the set spans $\mathbb{R}^3$. Conclusion: the set is a basis for $\mathbb{R}^3$.

2. Since this set contains the zero vector, it cannot be linearly independent, or it is linearly dependent. The set does not span $\mathbb{R}^3$ either. You can check that the vector $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ cannot be expressed as a linear combination of those vectors.

7. You can easily check the set is linearly independent. The set does not span $\mathbb{R}^3$ because $\begin{bmatrix} 15 \\ 0 \\ 15 \end{bmatrix}$ cannot be written as a linear combination of those two vectors.

10. The reduced echelon form of $A = \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -5 & 0 & 7 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$, so first of all we see that $\dim \mathrm{Nul} A = 2$, the number of nonpivot columns. The solution set of the system $A\vec{x} = \vec{0}$ is then $\left\{ \begin{bmatrix} 5x_3 - 7x_5 \\ 4x_3 - 6x_5 \\ x_3 \\ 3x_5 \\ x_5 \end{bmatrix} : x_3, x_5 \text{ are free} \right\} = \mathrm{Span} \left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$.

So $\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul$A$.

14. B is not in the *reduced* echelon form, but it is easy to see that B (and hence A) has three pivot columns: first, third, and fifth. Therefore $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$ forms a basis for Col$A$. You can find a basis for Nul$A$ using the same method as in **10**. If you got $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ \frac{7}{5} \\ 1 \\ 0 \end{bmatrix} \right\}$, then you did it right.

16 The problem asks you to find a basis for Col$A$, where A is the matrix having given vectors as columns. $A = \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, which shows that columns 1,2,3 are pivotal. So $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix} \right\}$ is a basis for the vector space..

19. Let $A = \begin{bmatrix} 4 & 1 & 7 \\ -3 & 9 & 11 \\ 7 & -2 & 6 \end{bmatrix}$, the matrix having $\vec{v}_1, \vec{v}_2, \vec{v}_3$ as columns. The equality $4\vec{v}_1 + 5\vec{v}_2 - 3\vec{v}_3 = \vec{0}$ shows that Nul$A \neq \{\vec{0}\}$, or $\dim \text{Nul}A \geq 1$. By the rank theorem, $\dim \text{Col}A \leq 2$. Now consider Col$A$. Since $\vec{v}_1$ and $\vec{v}_2$ are not scalar multiples of each other, the set $S = \{\vec{v}_1, \vec{v}_2\}$ is linearly independent, meaning $\dim \text{Col}A \geq 2$. We conclude that $\dim \text{Col}A = 2$ and Span$S$ is a 2-dimensional subspace of Col$A$. In order for this to happen, we must have Col$A = \text{Span}\{\vec{v}_1, \vec{v}_2\}$. Since we already showed that S is linearly independent, we see that S is a basis for Col$A = H$.

23. By the basis theorem.

24. By the basis theorem.