SOLUTIONS TO SELECTED PROBLEMS IN SECTION 5.3

2. $A^4 = PD^4P^{-1} = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{311}{16} & \frac{93}{8} \\ -\frac{465}{16} & -\frac{139}{8} \end{bmatrix}$

8. Let $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$, then $\det(A - \lambda I_2) = \begin{vmatrix} 5-\lambda & 1 \\ 0 & 5-\lambda \end{vmatrix} = (\lambda - 5)^2 = 0$ if and only if $\lambda = 5$. To find eigenvector(s) which correspond to 5, let's find Nul $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. It is easy to see that Nul $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Therefore it is impossible to find two linearly independent eigenvectors of A. Thus A is not diagonalizable.

14. Let A be the given matrix. $\begin{vmatrix} 4-\lambda & 0 & -2 \\ 2 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (4-\lambda)(5-\lambda)^2$. Let's find eigenvector(s) corresponding to $\lambda = 4$ first. Verify that Nul $\begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$. For $\lambda = 5$, we have Nul $\begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$. We see that the set $\left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent, and hence A is diagonalizable. Now let $P = \begin{bmatrix} -\frac{1}{2} & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $P^{-1}AP = D$, where $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

17. Let $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, then $\det(A - \lambda I_3) = \begin{vmatrix} 4-\lambda & 0 & 0 \\ 1 & 4-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = (4-\lambda)^2(5-\lambda)$ and hence we have two distinct eigenvalues 4 and 5. Note that 4 and 5 have algebraic multiplicities 2 and 1, respectively. Let's find the eigenspace corresponding to 4 which is given by Nul$(A - 4I_3)$. It is easy to see that Nul$(A - 4I_3) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ and this means that the geometric dimension of the eigenvalue 4 is just 1. Therefore, by Theorem 7.b in p.324, A is not diagonalizable.

19. Let $A = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$. The characteristic equation is now

$$\det \begin{bmatrix} 5-\lambda & -3 & 0 & 9 \\ 0 & 3-\lambda & 1 & -2 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix} = (5-\lambda)(3-\lambda)(2-\lambda)^2 = 0,$$

which has 5, 3 and 2 as solutions.

When $\lambda = 5$, $\begin{bmatrix} 5-\lambda & -3 & 0 & 9 \\ 0 & 3-\lambda & 1 & -2 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix}$ becomes $\begin{bmatrix} 0 & -3 & 0 & 9 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$ whose reduced echelon form is $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore the eigenspace corresponding to the eigenvalue $\lambda = 5$ is Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

When $\lambda = 3$, $\begin{bmatrix} 5-\lambda & -3 & 0 & 9 \\ 0 & 3-\lambda & 1 & -2 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix}$ becomes $\begin{bmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ whose reduced echelon form is $\begin{bmatrix} 0 & -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore the eigenspace corresponding to the eigenvalue $\lambda = 3$ is Span $\left\{ \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

When $\lambda = 2$, $\begin{bmatrix} 5-\lambda & -3 & 0 & 9 \\ 0 & 3-\lambda & 1 & -2 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{bmatrix}$ becomes $\begin{bmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ whose reduced echelon form is $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore the eigenspace corresponding to the eigenvalue

$\lambda = 2$ is Span $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$, a 2-dimensional space.

Now, for each eigenvalue of A, the algebraic multiplicity is the same as the geometric multiplicity, so we conclude that A is diagonalizable and one invertible matrix P which makes $P^{-1}AP$ diagonal is given by collecting all the eigenvectors. For example, you may take $P = \begin{bmatrix} 1 & \frac{3}{2} & -1 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ to get $P^{-1}AP = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

21. **a**. False. D must be a diagonal matrix.
b. True.
c. False. If we count multiplicities, then every matrix has n eigenvalues (so called the fundamental theorem of algebra).
d. False. Two concepts are irrelevant.

22. **a**. False. A is diagonalizable if and only if A has n linearly independent eigenvectors.
b. False. See Problem **14** above for a counter-example.
c. True. Compare corresponding columns.
d. False. See Problem **8** above.

23. Yes, by Theorem 7.b in p.324.

27. Clearly A^{-1} is invertible with $(A^{-1})^{-1} = A$. Since A is diagonalizable, we can find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Note that D must be invertible (Why?) and its inverse is another diagonal matrix (Can you describe D^{-1} in terms of diagonal entries of D?). Now $A^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$ and this shows that A is diagonalizable.

31. See Problem **8** above. Another example is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The latter is (probably) the most frequently cited example of non-diagonalizable matrix. If A were diagonalizable, since 1 is the only eigenvalue of A, A must be equal to $P \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} P^{-1} = PI_2P^{-1} = I_2$. Contradiction. You can apply this reasoning to Problem **8** above.