

SOLUTIONS TO SELECTED PROBLEMS IN SECTION 1.3

1. First note that the text book uses **bold** letters to denote vectors. We keep using letters with an arrow as we did in class.

$$\vec{u} + \vec{v} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \quad -2\vec{v} = (-2)\vec{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \quad \vec{u} - 2\vec{v} = \vec{u} + (-2)\vec{v} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

5. The vector equation is equivalent to the system

$$\begin{aligned} 6x_1 - 3x_2 &= 1 \\ -x_1 + 4x_2 &= -7, \\ 5x_1 &= -5 \end{aligned}$$

which in turn equivalent to the following matrix-vector representation:

$$\begin{bmatrix} 6 & -3 \\ -1 & 4 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}.$$

8. \vec{w} represents the fourth vertex of the parallelogram having $2\vec{v}$ and $-\vec{u}$ as its two incident sides. So $\vec{w} = 2\vec{v} - \vec{u}$ or $-\vec{u} + 2\vec{v}$. Similarly, $\vec{x} = 2\vec{v} - 2\vec{u}$. It appears \vec{y} is placed at the fourth vertex of the parallelogram having $-2\vec{u}$ and $3.5\vec{v}$ as its sides, so $\vec{y} = 3.5\vec{v} - 2\vec{u}$. Use a little bit of imagination (neither $-3\vec{u}$ nor $4\vec{v}$ is shown) to get $\vec{z} = -3\vec{u} + 4\vec{v}$.

11. Check whether the system associated with the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 5 & \vdots & 2 \\ -2 & 1 & -6 & \vdots & -1 \\ 0 & 2 & 8 & \vdots & 6 \end{array} \right]$$

is consistent or not. The matrix becomes

$$\left[\begin{array}{cccc|c} 1 & 0 & 5 & \vdots & 2 \\ -2 & 1 & -6 & \vdots & -1 \\ 0 & 2 & 8 & \vdots & 6 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 + 2R_1} \left[\begin{array}{cccc|c} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & 4 & \vdots & 3 \\ 0 & 2 & 8 & \vdots & 6 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \left[\begin{array}{cccc|c} 1 & 0 & 5 & \vdots & 2 \\ 0 & 1 & 4 & \vdots & 3 \\ 0 & 0 & 0 & \vdots & 0 \end{array} \right]$$

and the rightmost column is not pivotal. Thus the system is consistent and hence \vec{b} is a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$. In other words, $\vec{b} \in \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$. Coefficients? We have more than one triple of coefficients which expresses \vec{b} in terms of $\vec{a}_1, \vec{a}_2, \vec{a}_3$. For example, $\vec{b} = 2\vec{a}_1 + 3\vec{a}_2 + 0\vec{a}_3 = 2\vec{a}_2 + 3\vec{a}_2$. $\vec{b} = -3\vec{a}_1 - \vec{a}_2 + \vec{a}_3$ also works. Can you find all such triples? I will give the answer, you justify why:

$$\vec{b} = (2 - 5t)\vec{a}_1 + (3 - 4t)\vec{a}_2 + t\vec{a}_3.$$

Note that two previous triples correspond to $t = 0, 1$, respectively.

14. Note that \vec{b} is a linear combination of columns of A if and only if the system $A\vec{x} = \vec{b}$ is consistent. Since the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -2 & -6 & \vdots & 11 \\ 0 & 3 & 7 & \vdots & -5 \\ 1 & -2 & 5 & \vdots & 9 \end{array} \right]$$

reduces to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & \vdots & \frac{245}{33} \\ 0 & 1 & 0 & \vdots & -\frac{41}{33} \\ 0 & 0 & 1 & \vdots & -\frac{2}{11} \end{array} \right],$$

the system is consistent (the rightmost column is not pivotal) and therefore \vec{b} is a linear combination of the columns of A .

17. Since

$$\left[\begin{array}{ccc|c} 1 & -2 & \vdots & 4 \\ 4 & -3 & \vdots & 1 \\ -2 & 7 & \vdots & h \end{array} \right] \xrightarrow[\text{operations}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & \vdots & -2 \\ 0 & 1 & \vdots & -3 \\ 0 & 0 & \vdots & h+17 \end{array} \right],$$

h must equal -17 in order for the rightmost column not to be pivotal.

19. Note that $\vec{v}_2 = \frac{3}{2}\vec{v}_1$ and hence they are collinear. Therefore $\text{Span}\{\vec{v}_1, \vec{v}_2\}$ is the line containing \vec{v}_1 (and hence \vec{v}_2).

21. It suffices to show that the system associated with the following augmented matrix

$$\left[\begin{array}{cc|c} 2 & 2 & \vdots & h \\ -1 & 1 & \vdots & k \end{array} \right]$$

is consistent no matter what values we choose for h and k . After some proper row operations, the matrix becomes

$$\left[\begin{array}{cc|c} 2 & 2 & \vdots & h \\ -1 & 1 & \vdots & k \end{array} \right] \xrightarrow[\text{row operations}]{\text{many}} \left[\begin{array}{cc|c} 1 & 0 & \vdots & \frac{h-2k}{4} \\ 0 & 1 & \vdots & \frac{h+2k}{4} \end{array} \right]$$

and the rightmost column of the last matrix is not pivotal for any choice of h, k (We have two pivot columns, say column 1 and 2). This observation completes the proof.

24. a. True.

b. True. One can prove this using properties in p.32.

c. False. They all could be zero. That's why we have $\vec{0}$ in the span of any collection of vectors.

d. True. If both \vec{u} and \vec{v} are zero vectors, the statement is no longer true.

e. True. The system is consistent if and only if $\vec{b} \in \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

29. Total mass = $2+5+2+1=10$.

$$\begin{aligned}\text{Center of gravity } \vec{v} &= \frac{1}{10} \left(2 \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} + 2 \begin{bmatrix} -4 \\ -3 \\ -1 \end{bmatrix} + \begin{bmatrix} -9 \\ 8 \\ 6 \end{bmatrix} \right) \\ &= \frac{1}{10} \left(\begin{bmatrix} 10 \\ -8 \\ 6 \end{bmatrix} + \begin{bmatrix} 20 \\ 15 \\ -10 \end{bmatrix} + \begin{bmatrix} -8 \\ -6 \\ -2 \end{bmatrix} + \begin{bmatrix} -9 \\ 8 \\ 6 \end{bmatrix} \right) \\ &= \frac{1}{10} \begin{bmatrix} 13 \\ 9 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1.3 \\ .9 \\ 0 \end{bmatrix}.\end{aligned}$$