

SOLUTIONS TO SELECTED PROBLEMS IN SECTION 1.4

1. undefined.

2. undefined.

4. The product would be a 2×1 matrix, and it is

$$\begin{bmatrix} 8 \cdot 1 + 3 \cdot 1 + (-4) \cdot 1 \\ 5 \cdot 1 + 1 \cdot 1 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$

13. Yes. The augmented matrix associated with this problem is

$$\begin{bmatrix} 3 & -5 & \vdots & 0 \\ -2 & 6 & \vdots & 4 \\ 1 & 1 & \vdots & 4 \end{bmatrix}$$

which becomes

$$\begin{bmatrix} 1 & 0 & \vdots & \frac{5}{2} \\ 0 & 1 & \vdots & \frac{3}{2} \\ 0 & 0 & \vdots & 0 \end{bmatrix}$$

after some row operations. Since the rightmost column of the last matrix is not a pivot column, the system is consistent and hence \vec{u} in the span of columns of A .

14. No. $\vec{u} = \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$ is in $\text{Span} \left\{ \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix} \right\}$ if and only if there are scalars c_1, c_2, c_3 such that

$$\begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 8 \\ 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix},$$

which is equivalent to saying that the system

$$\begin{aligned} 5c_1 + 8c_2 + 7c_3 &= 2 \\ c_2 - c_3 &= -3 \\ c_1 + 3c_2 &= 2 \end{aligned}$$

is consistent. Now the augmented matrix

$$\begin{bmatrix} 5 & 8 & 7 & \vdots & 2 \\ 0 & 1 & -1 & \vdots & -3 \\ 1 & 3 & 0 & \vdots & 2 \end{bmatrix}$$

associated with the above system becomes

$$\begin{bmatrix} 1 & 0 & 3 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$$

and the rightmost column is pivotal. This tells us that the system is inconsistent and hence \vec{u} is not in the span of column vectors of A .

Remark In general, if $\vec{u}, \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ are n -vectors (having n entries), how can we check whether \vec{u} is in $\text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ or not? There is a shortcut (Imitate the solution above to see why it works): form an $n \times (k+1)$ matrix A whose columns consist of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ and \vec{u} . That is,

$$A = \begin{bmatrix} | & | & \cdots & | & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k & \vec{u} \\ | & | & & | & | \end{bmatrix}$$

(Caution: \vec{u} must be the rightmost column in A !). Obtain the reduced row echelon form of A . If the rightmost column of the reduced echelon matrix is pivotal, then $\vec{u} \notin \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$. If the rightmost column of the reduced echelon matrix is not pivotal, then $\vec{u} \in \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$.

15. The matrix

$$\begin{bmatrix} 2 & -1 & \vdots & b_1 \\ -6 & 3 & \vdots & b_2 \end{bmatrix}$$

reduces to

$$\begin{bmatrix} 2 & -1 & \vdots & b_1 \\ 0 & 0 & \vdots & b_2 + 3b_1 \end{bmatrix}.$$

If $b_2 + 3b_1 \neq 0$, then the rightmost column would be pivotal, meaning in this case the system $A\vec{x} = \vec{b}$ is inconsistent. So for $A\vec{x} = \vec{b}$ is consistent if and only if $b_2 + 3b_1 = 0$.

25. The equation

$$\begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix}$$

holds for c_1, c_2, c_3 if and only if equations

$$\begin{aligned} 4c_1 - 3c_2 + c_3 &= -7 \\ 5c_1 - 2c_2 + 5c_3 &= -3 \\ -6c_1 + 2c_2 - 3c_3 &= 10 \end{aligned} \quad (*)$$

hold for c_1, c_2, c_3 . Given equality

$$\begin{bmatrix} 4 & -3 & 1 \\ 5 & -2 & 5 \\ -6 & 2 & -3 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -3 \\ 10 \end{bmatrix}$$

tells us that three equations in (*) are all satisfied when we plug in $(-3, -1, 2)$, so we can take $(c_1, c_2, c_3) = (-3, -1, 2)$.