SOLUTIONS TO SELECTED PROBLEMS IN SECTION 1.6

5. Let's determine the coefficients x_1, x_2, x_3, x_4 in the chemical equation

$$x_1 \mathrm{B_2S_3} + x_2 \mathrm{H_2O} \longrightarrow x_3 \mathrm{H_3BO_3} + x_4 \mathrm{H_2S}.$$

Comparing the number of B: $2x_1 = x_3$,
Comparing the number of S: $3x_1 = x_4$,
Comparing the number of H: $2x_2 = 3x_3 + 2x_4$, and
Comparing the number of O: $x_2 = 3x_3$.

The augmented matrix associated with the system is

$$\left[\begin{array}{cccc:c} 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 0 & 2 & -3 & -2 & 0 \\ 0 & 1 & -3 & 0 & 0 \end{array}\right]$$

and it becomes

$$\left[\begin{array}{cccc:c} 1 & 0 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The solution set is now

$$\begin{cases} x_1 & = & \frac{1}{3}x_4 \\ x_2 & = & 2x_4 \\ x_3 & = & \frac{2}{3}x_4 \\ x_4 & & \text{is free} \end{cases}.$$

We need one final step. Since each coefficient must be a whole number, x_4 must be a multiple of 3. The simplest is of course 3, so we have

$$\begin{cases} x_1 & = & 1 \\ x_2 & = & 6 \\ x_3 & = & 2 \\ x_4 & = & 3 \end{cases}$$

and we can balance the chemical equation as follows:

$$\mathrm{B_2S_3} + 6\mathrm{H_2O} \longrightarrow 2\mathrm{H_3BO_3} + 3\mathrm{H_2S}.$$

12. (a) Consider the flow-in flow-out equation at each intersection:

At A: $x_1 = 40 + x_3 + x_4$,
At B: $200 = x_1 + x_2$,
At C: $x_2 + x_3 = 100 + x_5$, and

At D: $x_4 + x_5 = 60$.

The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & \vdots & 40 \\ 1 & 1 & 0 & 0 & 0 & \vdots & 200 \\ 0 & 1 & 1 & 0 & -1 & \vdots & 100 \\ 0 & 0 & 0 & 1 & 1 & \vdots & 60 \end{bmatrix}.$$

This matrix row reduces to

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & \vdots & 100 \\ 0 & 1 & 1 & 0 & -1 & \vdots & 100 \\ 0 & 0 & 0 & 1 & 1 & \vdots & 60 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

and hence the solution set is

$$\begin{cases} x_1 & = & 100 + x_3 - x_5 \\ x_2 & = & 100 - x_3 + x_5 \\ x_3 & & \text{is free} \\ x_4 & = & 60 - x_5 \\ x_5 & & \text{is free} \end{cases}.$$

Since each x_i is a positive whole number, we have to set some limitations. First of all, in order for x_4 to be nonnegative, we have $0 \leq x_5 \leq 60$. Second, both x_1 and x_2 have to be nonnegative and that requires $-100 \leq x_3 - x_5 \leq 100$, or equivalently, $|x_3 - x_5| \leq 100$.

(b) This time, we have just four variables, say x_1, x_2, x_3, and x_5. Again consider the flow-in flow-out equation at each intersection. That will give:

At A: $x_1 = 40 + x_3$,
At B: $200 = x_1 + x_2$,
At C: $x_2 + x_3 = 100 + x_5$, and
At D: $x_5 = 60$.

The corresponding augmented matrix is

$$\begin{bmatrix} 1 & 0 & -1 & 0 & \vdots & 40 \\ 1 & 1 & 0 & 0 & \vdots & 200 \\ 0 & 1 & 1 & -1 & \vdots & 100 \\ 0 & 0 & 0 & 1 & \vdots & 60 \end{bmatrix},$$

which row reduces to

$$\begin{bmatrix} 1 & 0 & -1 & 0 & \vdots & 40 \\ 0 & 1 & 1 & 0 & \vdots & 160 \\ 0 & 0 & 0 & 1 & \vdots & 60 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

The solution set is

$$\begin{cases} x_1 & = & 40 + x_3 \\ x_2 & = & 160 - x_3 \\ x_3 & & \text{is free} \\ x_5 & = & 60 \end{cases}.$$

Note that we must have $0 \leq x_3 \leq 160$, because x_2 should be nonnegative.

(c) If $x_4 = 0$, we can regard the road whose flow is x_4 as closed. In part (b), we had $x_1 = 40 + x_3$ with $0 \leq x_3 \leq 160$. Therefore, the minimum value of x_1 is 40. The maximum value of x_1 is 200.