SOLUTIONS TO SELECTED PROBLEMS IN SECTION 2.2

2. Let $A = \begin{bmatrix} 3 & 2 \\ 7 & 4 \end{bmatrix}$, then $\det A = 3 \cdot 4 - 2 \cdot 7 = -2 \neq 0$, so A is invertible. Indeed $A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{7}{2} & -\frac{3}{2} \end{bmatrix}$.

3. Let $A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$, then $\det A = -40 + 35 = -5 \neq 0$. Therefore A is invertible and $A^{-1} = \frac{1}{-5} \begin{bmatrix} -5 & -5 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{7}{5} & -\frac{8}{5} \end{bmatrix}$.

6. The system can be written as $A\vec{x} = \vec{b}$, where $A = \begin{bmatrix} 8 & 5 \\ -7 & -5 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}$. Since A is invertible, the unique solution of this system is given by $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 1 & 1 \\ -\frac{7}{5} & -\frac{8}{5} \end{bmatrix} \begin{bmatrix} -9 \\ 11 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$.

9. (a) True. However, even if you know only $AB = I$, you automatically have $BA = I$.
(b) False. AB is invertible, but the formula should be $(AB)^{-1} = B^{-1}A^{-1}$.
(c) False. $\det A = ad - bc$, not $ab - cd$.
(d) True. Actually for each $\vec{b}$, the system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$.
(e) True. I didn't mention elementary matrices in class, so you can safely ignore this problem.

14. Multiplying both sides of $(B - C)D = 0$ on the right by D^{-1}, we get

$$B - C = (B - C)I_n = (B - C)(DD^{-1}) = ((B - C)D)D^{-1} = 0D^{-1} = 0.$$

In other words, $B = C$.

19. First, assume that the equation has a solution, say X. Multiplying both sides of the given equation by C, on the left, and by B, on the right, we get

$$CC^{-1}(A + X)B^{-1}B = CI_nB,$$

which reduces to $A + X = CB$, or equivalently $X = CB - A$. What we have shown so far is: if the equation has a solution, it must be $CB - A$. Now we claim that $CB - A$ really is a solution. How can we prove this? Just plug this back to the original equation, then it is not hard to see that the equality holds.

31. Since $\begin{bmatrix} 1 & 0 & -2 & \vdots & 1 & 0 & 0 \\ -3 & 1 & 4 & \vdots & 0 & 1 & 0 \\ 2 & -3 & 4 & \vdots & 0 & 0 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 0 & \vdots & 8 & 3 & 1 \\ 0 & 1 & 0 & \vdots & 10 & 4 & 1 \\ 0 & 0 & 1 & \vdots & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$, the given matrix is invertible with the inverse $\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$.

32. Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -4 \end{bmatrix}$. First we form a bigger matrix $\begin{bmatrix} 1 & -2 & 1 & \vdots & 1 & 0 & 0 \\ 4 & -7 & 3 & \vdots & 0 & 1 & 0 \\ -2 & 6 & -4 & \vdots & 0 & 0 & 1 \end{bmatrix}$ and then *try* to transform A into I_3. Replace R_2 by $R_2 - 4R_1$ and R_3 by $R_3 + 2R_1$ to get $\begin{bmatrix} 1 & -2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & -1 & \vdots & -4 & 1 & 0 \\ 0 & 2 & -2 & \vdots & 2 & 0 & 1 \end{bmatrix}$. Here, replace R_3 by $R_3 - 2R_2$, then we have $\begin{bmatrix} 1 & -2 & 1 & \vdots & 1 & 0 & 0 \\ 0 & 1 & -1 & \vdots & -4 & 1 & 0 \\ 0 & 0 & 0 & \vdots & 10 & -2 & 1 \end{bmatrix}$. Now we see that no matter what row operations we use, we cannot get I_3 on the left. Thus we conclude that A is not invertible. Note that columns of A are linearly dependent, because A is not invertible. You can also directly check linear dependence of columns: $\vec{v}_2 = -\vec{v}_1 - \vec{v}_3$, where $\vec{v}_i$ $(i = 1, 2, 3)$ denotes the i^{th} column of A.