SOLUTIONS TO SELECTED PROBLEMS IN SECTION 3.2

1. If you interchange two rows (R_1 and R_2 in this problem), and then you have to reverse the sign of the original determinant.

2. If A' is obtained by multiplying one row (R_1) of A by a scalar k (2), then $\det A' = k \cdot \det A$.

4. Note that we used $R_3 \mapsto R_3 - 3R_1$ to get the second matrix. Replacement operations do not change the determinant.

15. $5 \cdot 7 = 35$.

16. $3 \cdot 7 = 21$.

17. -7.

18. 7, since the matrix is obtained by interchanging R_1 and R_2 in the matrix in the problem **17** above.

19. 14. Starting from the original matrix, apply $R_2 \mapsto 2R_1$ and then $R_2 \mapsto R_2 + R_1$ to get the matrix. The first operation multiplies the original determinant by 2, giving 14. The second changes nothing.

20. 7. To get the matrix, we used $R_1 \mapsto R_1 + R_2$, and the determinant is not affected by replacement operations.

22. Let's use the expansion along the first row. Then

$$\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = 5 \begin{vmatrix} -3 & -2 \\ 5 & 3 \end{vmatrix} - \begin{vmatrix} 1 & -3 \\ 0 & 5 \end{vmatrix} = 0,$$

and hence the given matrix is not invertible. Note that the columns of the matrix are linearly dependent. Do you see that $\vec{v}_1$ is in the span of $\vec{v}_2$ and $\vec{v}_3$, where $\vec{v}_i$ denotes the i^{th} column of the matrix? Actually, $\vec{v}_1 = 3\vec{v}_2 - 5\vec{v}_3$.

26. Form the matrix consisting of given vectors, say

$$A = \begin{bmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{bmatrix}.$$

Remember, the columns are linearly independent iff A is invertible. So let's calculate $\det A$. The expansion down the fourth column would be the best choice. Actually,

$$\begin{vmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -3 \end{vmatrix} = -3 \begin{vmatrix} 3 & 2 & -2 \\ 5 & -6 & -1 \\ -6 & 0 & 3 \end{vmatrix} = 0.$$

Check this for yourself.

27. **a**. True.
b. Ignore this. For those who are interested: it is False.
c. True.
d. False. Put $A = B = I_n$ for a counterexample.

28. **a**. True.
b. False. If A were a lower or upper triangular matrix, then it would be true.
c. False. See the problem **26** for a counterexample.
d. False. $\det A^T = \det A$.

29. Note that $\det B^5 = (\det B)^5$. Now check that $\det B = -2$ for yourself. So $\det B^5 = (-2)^5 = -32$.

31. First note that the formula makes sense since the denominator $\det A$ does not vanish provided A is invertible. Since $AA^{-1} = I_n$, taking the determinant of both sides we get

$$\det A \cdot \det A^{-1} = \det AA^{-1} = \det I_n = 1$$

or $\det A^{-1} = \frac{1}{\det A}$.

32. Apply Theorem 3. c. in p.192 n times, then you get $\det rA = r^n \det A$.

33. $\det AB = \det A \cdot \det B = \det B \cdot \det A = \det BA$, where the second equality from the commutativity of the product of real *numbers*.

34. $\det PAP^{-1} = \det P \cdot \det A \cdot \det P^{-1} \overset{\text{problem}\mathbf{31}}{=} \det P \cdot \det A \cdot \frac{1}{\det P} = \det A$.

35. First, $\det U^T \cdot \det U = \det U^T U = \det I_n = 1$. Since $\det U^T = \det U$ in general, actually we have $(\det U)^2 = 1$ or $\det U = \pm 1$.
Remark: When $U^T U = I_n$, U is called an orthogonal matrix. Examples: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Orthogonal matrices play an important in the diagonalization of real symmetric matrices, though it is beyond the scope of this course.

36. It suffices to show that $\det A = 0$. Indeed, since $\det A^4 = (\det A)^4$, it follows that $(\det A)^4 = 0$. This can happen only if $\det A = 0$.

39. **a**. $4 \cdot (-3) = -12$.
b. $5^3 \cdot 4 = 500$.
c. -3.
d. $\frac{1}{4}$.
e. $4^3 = 64$.