SOLUTIONS TO SELECTED PROBLEMS IN SECTION 4.5

3. First note that it is a subspace of $\mathbb{R}^4$, because it is the span of a collection of vectors. Now $\left\{ \begin{bmatrix} 2c \\ a-b \\ b-3c \\ a+2b \end{bmatrix} : a,b,c \text{ in } \mathbb{R} \right\} = \left\{ a\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix} + c\begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} : a,b,c \text{ in } \mathbb{R} \right\}$ and it can be written as Span$S$, where $S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix} \right\}$. Note that S is linearly independent, so S is a basis for Span$S$ and in particular the dimension of Span$S$ is 3.

12. First note that the given vector space in the problem is nothing but $\operatorname{Col} A$, where $A = \begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix}$. Therefore the desired dimension is the rank of A. Since the reduced echelon form of A is $\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, the dimension of the vector space is 3.

14. $\dim \operatorname{Col} A = \operatorname{rank} A = 3$ and hence $\dim \operatorname{Nul} A = 6 - 3 = 3$ by the rank theorem.

19. **a**. True.
b. False. A plane in $\mathbb{R}^3$ *containing the origin* is a subspace of $\mathbb{R}^3$.
c. False. $\dim \mathbb{P}_4 = 5$.
d. False. $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is linearly independent in $\mathbb{R}^3$, but S does not span $\mathbb{R}^3$, so S is not a basis for $\mathbb{R}^3$.
e. True. Since $\{\vec{v}_1, \cdots, \vec{v}_p\}$ spans V, $\dim V$ is at most p. Now T contains too many elements to be linearly independent.

20. **a**. False. $\mathbb{R}^2$ and $\mathbb{R}^3$ are totally different as sets. So one cannot be a subspace of the other.
b. False. The number of *free* variables in the equation $A\vec{x} = \vec{0}$ equals the dimension of Nul$A$.
c. False. Infinitely many vectors may generate a just one-dimensional space!
d. False. Let T be a basis for V and let $S = T \cup \{\vec{0}\}$ (S is different from T. Why?). Then S spans V, but clearly S is linearly dependent because it contains the zero vector.

e. True. In general, if H is a subspace of a finite dimensional vector space V and $\dim H = \dim V$, then $H = V$.

21. By the basis theorem, it suffices to show that the set $S = \{1, 2t, -2+4t^2, -12t+8t^3\}$ is linearly independent. Suppose $c_1 \cdot 1 + c_2 \cdot 2t + c_3 \cdot (-2+4t^2) + c_4 \cdot (-12t+8t^3) = z(t)$, the zero polynomial. In other words, suppose $c_1 - 2c_3 + (2c_2 - 12c_4)t + 4c_3t^2 + 8c_4t^3$ is identically zero. Since we know that the set $\{1, t, t^2, t^3\}$ is linearly independent, it follows that $c_1 - 2c_3 = 2c_2 - 12c_4 = 4c_3 = 8c_4 = 0$, or, $c_1 = c_2 = c_3 = c_4 = 0$. Thus S is linearly independent.

25. Assume that $\text{Span}S = V$. Suppose $S = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k\}$ with $k < n$. If S is linearly dependent, then using the spanning set theorem we can remove a vector in S and still keep the same span. Iterating this, we eventually have a linearly independent set $T \subset S$ such that $\text{Span}T = \text{Span}S = V$, which would mean that T is a basis of V. Now $n = \dim V =$ the number of vectors in $T \leq$ the number of vectors in $S = k < n$. This contradiction shows that $\text{Span}S$ cannot be V.

29. **a**. True. Mimic the reasoning in **25**.
b. True. Consider $H = \text{Span}\{\vec{v}_1, \cdots, \vec{v}_p\}$, then $p = \dim H \leq \dim V$.
c. True. Let S be a basis for V. Consider the set $T = S \cup \{\vec{0}\}$, then T has exactly $p+1$ elements and $\text{Span}T = V$. Of course, other answers are possible, but whatever your $(p+1)$-element spanning set is, the set must be linearly dependent (Why?).

30. **a**. False. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}$ is a linearly dependent subset of $\mathbb{R}^3$, but $\dim \mathbb{R}^3 = 3$.
b. True. Suppose for contradiction that $\dim V = n \leq p$, then we can find a basis $S = \{\vec{v}_1, \cdots, \vec{v}_n\}$ for V. In particular $\text{Span}S = V$. Adding suitable $p - n$ vectors (for an example, we may add $2\vec{v}_1, 3\vec{v}_1, \cdots, (p-n+1)\vec{v}_1$) to S, we have a set T of p elements with $\text{Span}T = V$. This is a contradiction and therefore $\dim V > p$.
c. False. The example in **a** works.