Exam 2

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Question 1. (10 points) Compute the determinants of the following matrices:

(a) $\begin{pmatrix} 1 & 3 & 3 \\ -2 & -4 & -4 \\ 1 & 1 & 4 \end{pmatrix}$

$$\begin{vmatrix} 1 & 3 & 3 \\ -2 & -4 & -4 \\ 1 & 1 & 4 \end{vmatrix} = -2 \begin{vmatrix} 1 & 3 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 4 \end{vmatrix} = -2 \begin{vmatrix} 1 & 3 & 3 \\ 0 & -1 & -1 \\ 0 & -2 & 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} = 6$$

(b) $\begin{pmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{pmatrix}$

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & b+c \\ 0 & b-a & a-b \\ 0 & c-a & a-c \end{vmatrix} \overset{C2+C3}{=} \begin{vmatrix} 1 & a+b+c & b+c \\ 0 & 0 & a-b \\ 0 & 0 & a-c \end{vmatrix}$$

$$= 0$$

Or

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} \overset{C2+C3}{=} \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & a+b+c & a+c \\ 1 & a+b+c & b+a \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & a+c \\ 1 & 1 & b+a \end{vmatrix} (a+b+c)$$

$$= 0$$

Question 2. (6 points) Use Cramer's rule to solve the following system of linear equations.

$$\begin{cases} sx_1 + x_2 &= s \\ x_1 + sx_2 &= 2 \end{cases}$$

If there are values for s that make the system have no solutions or infinitely many solutions, list these.

$$A = \begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix} \qquad |A| = s^2 - 1$$

$$A_1(\vec{b}) = \begin{pmatrix} s & 1 \\ 2 & s \end{pmatrix} \quad |A_1(\vec{b})| = s^2 - 2$$

$$A_2(\vec{b}) = \begin{pmatrix} s & s \\ 1 & 2 \end{pmatrix} = s$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{s^2-2}{s^2-1} \\ \frac{s}{s^2-1} \end{pmatrix}$$

No solution with $s = \pm 1$

Question 3. (9 points) For the following matrix A,

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 2 & 1 & 1 \end{pmatrix}$$

(a) Find vector(s) $\vec{v}_1, \ldots, \vec{v}_n$ such that their linear combinations

$$a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$

are exactly the solutions to $A\vec{x} = \vec{0}$.

Solutions to $A\vec{x} = \vec{0}$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 1 & 3 & 3 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$\vec{V}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ is the only vector needed

(b) Find a matrix C such that the solutions of $C\vec{x} = \vec{0}$ are exactly the linear combinations of the columns of A.

linear combinations of the columns of A: vectors $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ with:

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 1 & 3 & 3 & b_2 \\ 2 & 1 & 1 & b_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 1 & 1 & b_2 - b_1 \\ 0 & -3 & -3 & b_3 - 2b_1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & 1 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - 2b_1 + 3b_2 - 3b_1 \end{array}\right)$$

$$\cancel{\begin{pmatrix} -3 & +1 & 1 \end{pmatrix}} \quad \begin{pmatrix} -5 & 3 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0$$

$$C = \begin{pmatrix} -5 & 3 & 1 \end{pmatrix}$$

Question 4. (15 points) Consider these two matrices A and B.

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 & 1 \\ 1 & 2 & 1 & -1 & -3 \\ 2 & 4 & 2 & -2 & -6 \\ -1 & -2 & -3 & 1 & 7 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

You can use that A can be row-reduced to B. Find all of the following.

(a) $\dim Col(A), \dim Nul(A), \operatorname{rank} A$

dim Col A = 2
dim Nul A = 3
rank A = 2

(b) A basis for $Col(A)$, $Nul(A)$, $Row(A)$.

basis Col(A): $\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ -3 \end{pmatrix} \right\}$

basis row(A) = $\{ (1 \ 2 \ 0 \ -1 \ -1), (0 \ 0 \ 1 \ 0 \ -2) \}$

basis Nul A: $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$

(c) The dimension of the following vector spaces:

$$V = \left\{\vec{b} \text{ in } \mathbb{R}^4 \text{ with } A\vec{x} = \vec{b} \text{ has a solution.}\right\} \quad W = \left\{\vec{a} \text{ in } \mathbb{R}^5 \text{ with } A\vec{a} = \vec{0}\right\}$$

$V = \text{Col } A$

$\dim V = 2$

$W = \text{Nul}(A)$

$\dim W = 3$

(d) Find two linear equations in five unknowns that together adequately describe the solutions to $A\vec{x} = \vec{0}$.

need the basis of row A.

$$x_1 + 2x_2 - x_4 - x_5 = 0$$
$$x_3 - 2x_5 = 0$$

Question 5. (9 points) True or False? For each true statement, provide a brief explanation why it's true. For each false statement, give a specific counterexample.

(a) When two matrices A, B are row equivalent, they must have the same determinant.

False, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is row equivalent to $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(b) When $Span\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \mathbb{R}^n$, then n must be 4.

False, $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} = \mathbb{R}^2$

(c) For all square matrices A, $\det(A^T A) \geq 0$.

True,
$$\begin{aligned} \det(A^T A) &= \det A^T \cdot \det A \\ &= \det A \cdot \det A \\ &= (\det A)^2 \geq 0 \end{aligned}$$

Question 6. (9 points) Find an example of:

(a) A subspace of $\mathbb{R}^2$ that does not have a basis. Hint: there is only one.

$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

(b) A basis of $\mathbb{R}^3$ containing the vector $\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$.

$$\left\{ \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$ or many others

(c) An *invertible* matrix A such that $\det(-A) = \det A$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

Any invertible matrix with an even number of columns is good.

Question 7. (8 points) Consider the following vector space in $\mathbb{R}^4$.

$$V = \left\{ \begin{pmatrix} a-2b+5c \\ 2a+5b-8c \\ -a-4b+7c \\ 3a+b+c \end{pmatrix} \text{ with } a,b,c \text{ in } \mathbb{R} \right\}$$

Find a basis for V. What is the dimension of V?

$$V = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \\ -4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -8 \\ 7 \\ 1 \end{pmatrix} \right\}$$

Do these 3 form a basis?

$$\begin{pmatrix} 1 & -2 & 5 \\ 2 & 5 & -8 \\ -1 & -4 & 7 \\ 3 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 5 \\ 0 & 9 & -18 \\ 0 & -6 & 12 \\ 0 & 7 & -14 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

No! the vectors are linear ~~in~~dependent

a basis of V is $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \\ -4 \\ 1 \end{pmatrix} \right\}$

$\dim V = 2$

Question 8. (9 points) For each of these questions, explain your answer.

(a) Suppose a nonhomogeneous system of six linear equations in eight unknowns has a solution, with two free variables. Is it possible to pick a different target vector (i.e. right–hand side of constants) to make the new system inconsistent?

$A\vec{x} = \vec{b}$ $\quad$ A is 6×8

$\dim \mathrm{Nul}\, A = 2$

$\mathrm{rank}\, A = 8 - 2 = 6$

$\dim \mathrm{Col}\, A = 6$

$\Rightarrow \mathrm{Col}\, A = \mathbb{R}^6$

No, every $\vec{b}$ makes $A\vec{x} = \vec{b}$ consistent.

(b) Suppose a nonhomogeneous system of nine linear equations in ten unknowns has a solution for all possible target vectors. Is it possible to find two nonzero solutions of the associated homogeneous system that are *not* multiples of each other?

$A\vec{x} = \vec{b}$, A is 9×10

$\dim \mathrm{Col}\, A = 9$, $\mathrm{rank}\, A = 9$

$\dim \mathrm{Nul}\, A = 10 - 9 = 1$

No, all solutions to $A\vec{x} = \vec{0}$ are multiples of one another

(c) Suppose a nonhomogeneous system of five linear equations in six unknowns is inconsistent. What is the minimal dimension of the solution space of the associated homogeneous system?

$A\vec{x} = \vec{b}$ $\quad$ $\mathrm{rank}\, A < 5$ i.e. 4 or lower

A is 5×6

$\underset{4 \text{ or less}}{\mathrm{rank}\, A} + \underset{2 \text{ or more}}{\dim \mathrm{Nul}\, A} = 6$

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