Practice Exam 1

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November 29, 2008

Question 1. (8 points) Compute the following products of matrices.

a) $\begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -9 \\ 8 \end{pmatrix} = \begin{pmatrix} -27 \\ -18 \\ 0 \\ -9 \end{pmatrix} + \begin{pmatrix} 8 \\ 8 \\ -8 \\ 8 \end{pmatrix} = \begin{pmatrix} -19 \\ -10 \\ -8 \\ -1 \end{pmatrix}$

b) $\begin{pmatrix} 4 & 5 & 1 \\ 3 & 7 & 1 \end{pmatrix} \begin{pmatrix} -1/3 & 1 \\ 1/3 & 1 \\ -1/3 & -9 \end{pmatrix} = \begin{pmatrix} -4/3 + 5/3 - 1/3 & 4+5-9 \\ -1 + 7/3 - 1/3 & 3+7-9 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

Question 2. (6 points) Find the inverse of the following matrix.

$$\begin{pmatrix} 1 & -4 & 3 \\ 1 & -3 & 5 \\ -2 & 7 & -7 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -4 & 3 & 1 & 0 & 0 \\ 1 & -3 & 5 & 0 & 1 & 0 \\ -2 & 7 & -7 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R3+2R1]{R2-R1} \left(\begin{array}{ccc|ccc} 1 & -4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & -1 & -1 & 2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R3+R2} \left(\begin{array}{ccc|ccc} 1 & -4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[R1-3R3]{R2-2R3} \left(\begin{array}{ccc|ccc} 1 & -4 & 0 & -2 & -3 & -3 \\ 0 & 1 & 0 & -3 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right) \xrightarrow{R1+4R2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -14 & -7 & -11 \\ 0 & 1 & 0 & -3 & -1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

1 point true/False
2 points explanation

Question 3. (12 points) True or False? For each true statement, provide a brief explanation why it's true. For each false statement, give a specific counterexample.

a) The zero vector is a solution to every consistent linear system.

False : $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ does not have $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as a solution

b) Every linear system has the same number of solutions as its associated homogeneous system.

False : $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ has no solutions

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has infinitely many solutions

c) If $\{v_1, v_2, v_3, v_4\}$ are linearly *independent*, then $\{v_1, v_3, v_4\}$ are also linearly independent.

True : if there is a linear dependence relation $c_1 \vec{v}_1 + c_3 \vec{v}_3 + c_4 \vec{v}_4 = \vec{0}$,

then also $c_1 \vec{v}_1 + 0 \vec{v}_2 + c_3 \vec{v}_3 + c_4 \vec{v}_4 = 0$, and the 4 vectors would be linearly dependent.

d) If the columns of a square matrix A span $\mathbb{R}^n$, then the equation $A^2\vec{x} = \vec{0}$ has at least 7 solutions.

False: counterexample $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$A^2 \vec{x} = \vec{0}$ has 1 solution

(the columns of A span $\mathbb{R}^n \Rightarrow$ A and A^2 are invertible)

Question 4. (9 points) Give an example of:

a) A matrix with linearly independent columns that *cannot* be row reduced to the identity matrix.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$$

b) A square matrix which does not have any zero entries and is not invertible.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

c) A pair of 2×2 matrices that commute with each other.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$$

or

or ...

$$\begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$$

Question 5. (6 points) In this question, A, B, X are all square matrices of the same size, and let

$$(A - AX)^{-1} = X^{-1}B$$

(When $^{-1}$ is added to a matrix, it means the inverse exists.) Explain why B is invertible.

$$B = X X^{-1} B = X (A - AX)^{-1}$$

So B is the product of invertible matrices.

Question 6. (4 points) Let C be a 4×4 matrix, and let the two first columns of C be

$$\begin{pmatrix} 2 \\ 1 \\ 3 \\ -4 \end{pmatrix} \begin{pmatrix} -4 \\ -2 \\ -3 \\ 8 \end{pmatrix}$$

(You don't know anything about the two other columns of C.) Find three different solutions to $C\vec{x} = \vec{0}$.

$$\vec{V}_2 = -2\vec{V}_1 \implies 2\vec{V}_1 + \vec{V}_2 = \vec{0}$$

so $C \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \vec{0}$

other solutions: $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -8 \\ -4 \\ 0 \\ 0 \end{pmatrix}, \ldots$

Question 7. (12 points) Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 8 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 10 \\ 2 \end{pmatrix}$.

a) Show that $\vec{b}$ can be written as a linear combination of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ by finding a set of weights of that linear combination.

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 8 & 2 & 4 & 10 \\ 3 & 2 & 2 & 2 \end{array}\right) \xrightarrow[R3-3R1]{R2-8R1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 10 & 4 & -14 \\ 0 & 5 & \cancel{4} \; 2 & -7 \end{array}\right)$$

$$\xrightarrow{R3-\frac{1}{2}R2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 10 & 4 & -14 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - x_2 = 3$

$10x_2 + 4x_3 = -14$ pick $x_2 = -1$

$x_3 = -1$

$x_1 = 2$

$\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ are weights

b) Are there any vectors in $\mathbb{R}^3$ that cannot be written as a linear combination of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$?

Yes: $\begin{pmatrix} 1 & -1 & 0 \\ 8 & 2 & 4 \\ 3 & 2 & 2 \end{pmatrix}$ is <u>not</u> invertible (see part (a)) so its columns <u>do not</u> span $\mathbb{R}^3$

Question 8. (9 points) Find *all* values for h, k such that the system below is:

$$\begin{aligned} x_1 + 5x_2 &= 3 \\ -3x_1 + hx_2 &= k \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 5 & 3 \\ -3 & h & k \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 5 & 3 \\ 0 & h+15 & k+9 \end{array}\right)$$

a) underdetermined.

infinitely many solutions

$$\begin{cases} h+15=0 \\ k+9=0 \end{cases} \qquad \boxed{\begin{aligned} h&=-15 \\ k&=-9 \end{aligned}}$$

b) uniquely determined.

1 solution : pivot position in column 2

$h+15 \neq 0$ $\boxed{h \neq -15, \text{ any } k}$

c) overdetermined.

no solutions : pivot position in column 3

$h+15=0$, $k+9 \neq 0$

$$\boxed{\begin{aligned} h&=-15 \\ k&\neq -9 \end{aligned}}$$

Question 9. (9 points) Determine whether the following matrices are invertible.
Do not calculate inverses. Explain your answer.

a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$ invertible : pivot position in each column

b) $\begin{pmatrix} 1 & 3 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{R2-2R3} \begin{pmatrix} 1 & 3 & 9 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$ not invertible. (zero row)

c) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 8 \end{pmatrix}$ not invertible : not a square matrix.