Exam 1

NAME: SOLUTIONS

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When solving a system of linear equations, use row-reduction.

Question 1. (8 points) Compute the following products of matrices.

a) $\begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 1 & -1 \\ 1 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ -8 \\ 22 \\ 10 \end{pmatrix}$

b) $\begin{pmatrix} 2 & 1 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Question 2. (6 points) Find the inverse of the following matrix.

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & -2 & 3 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -2 & 3 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R3+R1]{R2+R1} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R2 \leftrightarrow R3} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & -3 & 2 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R3+3R2} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & -1 \\ 0 & 0 & 5 & -2 & 1 & -3 \end{array}\right)$$

$$\begin{array}{c} \tfrac{1}{5}R3 \\ R2-R3 \\ R1+R3 \end{array} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 3/5 & 1/5 & -3/5 \\ 0 & 1 & 0 & -3/5 & -1/5 & -2/5 \\ 0 & 0 & 1 & -2/5 & 1/5 & -3/5 \end{array}\right) \xrightarrow{R1+R2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3/5 & -1/5 & -2/5 \\ 0 & 0 & 1 & -2/5 & 1/5 & -3/5 \end{array}\right)$$

Answer: $\begin{pmatrix} 0 & 0 & -1 \\ -3/5 & -1/5 & -2/5 \\ -2/5 & 1/5 & -3/5 \end{pmatrix}$

Question 3. (4 points) Find the transpose of the following matrix:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 3 & -3 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & -3 & 0 \end{pmatrix}$$

Question 4. (9 points) True or False? For each true statement, provide a brief explanation why it's true. For each false statement, give a specific counterexample.

a) A linear system with more variables than equations is always consistent.

False : $\begin{pmatrix}1&1&1&1\\1&1&1&1\end{pmatrix}\vec{x} = \begin{pmatrix}2\\3\end{pmatrix}$

b) The sentence '$\vec{a}$ is a solution of the system $A\vec{x} = \vec{b}$.' means that $\vec{a}$ is a linear combination of the columns of A.

False : $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\vec{x} = \begin{pmatrix}2\\2\\2\\2\end{pmatrix}$ has solution 2

c) Let A be a $m \times n$ matrix, and let B be a $n \times m$ matrix such that $AB = I_m$. Then it is certain that $m = n$ and that B is the inverse of A.

False : question 1b :

$$\begin{pmatrix}2&1&4\\1&1&1\end{pmatrix}\begin{pmatrix}1&2\\-1&0\\0&-1\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Question 5. (9 points) Give an example of:

a) A matrix the columns of which span $\mathbb{R}^2$, but is *not* invertible.

$$\begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

b) A square matrix which does not have any zero entries and is not invertible.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

c) A pair of 2×2 matrices that commute with each other.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ \& \ \begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$$

Question 6. (14 points) Let $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$.

a) Show that $\vec{b}$ can be written as a linear combination of $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ by finding a set of weights of that linear combination.

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ -1 & 1 & -1 & 3 \\ 1 & 1 & 3 & -1 \end{array}\right) \xrightarrow[R3-R1]{R2+R1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 3 & 3 & 3 \\ 0 & -1 & -1 & -1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R1-2R2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

example weights: $(-2, 1, 0)$

b) What is the signature of the matrix $(\,\vec{v}_1 \quad \vec{v}_2 \quad v_3\,)$? Use the signature to find out whether every vector $\mathbb{R}^3$ is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Signature $\begin{pmatrix} \square & * & * \\ 0 & \square & * \\ 0 & 0 & 0 \end{pmatrix}$

No pivot position in 3^{rd} row $\rightarrow$

Not every vector in $\mathbb{R}^3$ is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$

Question 7. (9 points) Find *all* values for h, k such that the system below is:

$$\begin{aligned} x_1 + x_2 &= 3 \\ -hx_1 + 3x_2 &= k \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ -h & 3 & k \end{array}\right) \xrightarrow{R2 + hR1} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 3+h & k+3h \end{array}\right)$$

a) underdetermined.

$3 + h = 0$

$k + 3h = 0$

$h = -3$, $k = 9$

b) uniquely determined.

$h \neq -3$

any k

c) overdetermined.

$h = -3$

$k \neq 9$

Question 8. (9 points) Determine whether the following matrices are invertible. *Do not* calculate inverses. Explain your answer.

a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix}$ invertible, pivot position in each column

b) $\begin{pmatrix} 1 & 3 & 9 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 9 \\ 2 & 4 & 6 \\ 0 & 0 & 0 \end{pmatrix}$ not invertible

c) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & -2 & 8 \end{pmatrix}$ not invertible, not a square matrix

Question 9. (7 points) Let E be a 4×5 matrix such that

$$E \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 1 \\ -6 & -3 & 1 \\ -6 & -3 & 1 \end{pmatrix}$$

And notice that

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 1 \\ -6 & -3 & 1 \\ -6 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 2 \\ 4 \\ -6 \\ -6 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 2 \\ -3 \\ -3 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

Then:

a) What is the first column of E? What is the second column of E?

first: $\begin{pmatrix} 2 \\ 4 \\ -6 \\ -6 \end{pmatrix}$ second $\begin{pmatrix} 1 \\ 2 \\ -3 \\ -3 \end{pmatrix}$

b) Find two solutions to $E\vec{x} = \vec{0}$.

$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ since $2\vec{V_2} = \vec{V_1}$

c) Find two solutions to $E\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$