Practice Problems 2

2)
$$\begin{vmatrix} a-d & b-e & c-f \\ d-g & e-h & f-i \\ g-a & h-b & i-c \end{vmatrix} \overset{R2+R1}{=} \begin{vmatrix} a-d & b-e & c-f \\ a-g & b-h & c-i \\ g-a & h-b & i-c \end{vmatrix}$$

$$\overset{R3+R2}{=} \begin{vmatrix} a-d & b-e & c-f \\ a-g & b-h & c-i \\ 0 & 0 & 0 \end{vmatrix}$$

$$= 0$$

3) a) False : $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}$

b) True : theorem

c) False : $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

d) False : $\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$ not $-\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$

e) False : the coefficient matrix has to be invertible.
Counterexample : $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

6) V_1 is not a vector space:

$\begin{pmatrix} -2 \\ -1/2 \end{pmatrix}$ and $\begin{pmatrix} 1/2 \\ 2 \end{pmatrix}$ are in V_1 but their sum

$\begin{pmatrix} -3/2 \\ 3/2 \end{pmatrix}$ is not

V_2 is a vector space since it is $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$

(Since each vector in V_2 is a linear combination of these vectors with weights a, b and c, resp.)

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \xrightarrow{R3+R1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R3+R2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

So the first two vectors (the columns with a pivot position) form a basis for V_2

V_3: polynomials $p(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$
$p(0) = f = 0$

$ax^5 + bx^4 + cx^3 + dx^2 + ex$: linear combination of

x^5, x^4, x^3, x^2, x, and these are linearly independent $\to$ basis

8) a) $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & -1 \end{pmatrix}$ has 3 pivot positions.

Since $\mathbb{R}^3$ has dimension 3 (basis $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$) and these vectors are linearly independent, they form a basis for $\mathbb{R}^3$

b) $\begin{pmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{pmatrix} \xrightarrow{R3+2R1} \begin{pmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ 0 & 2 & -5 \end{pmatrix}$

has only 2 pivot positions.

It is not a basis, since the vectors are linearly dependent. The vectors also <u>do not</u> span $\mathbb{R}^3$ since you need 3 pivot positions in $A\vec{x} = \vec{b}$ for the columns of A to span ~~$\mathbb{R}$~~ a 3-dimensional space.

(If these vectors did span $\mathbb{R}^3$, and there were 3, then it would be a basis and they'd have to be linearly independent.)

c) $\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \xrightarrow[R3-3R1]{R2-2R1} \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \xrightarrow{R3-2R2} \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$

Not a basis, similar to (b)

9 basis Col A: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -5 \\ -5 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 5 \\ -2 \end{pmatrix} \right\}$

basis Row A: $\{ (1\ 2\ 0\ 4\ 5), (0\ 0\ 5\ -7\ 8), (0\ 0\ 0\ 0\ -9) \}$

for $A\vec{x} = 0$:

$$\begin{aligned} x_1 &= -2x_2 - 4x_4 \\ x_2 &= x_2 \\ x_3 &= \tfrac{7}{5} x_4 \\ x_4 &= x_4 \\ x_5 &= 0 \end{aligned}$$

basis for Nul A: $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{pmatrix} \right\}$

11. V has a basis

$\{1, x, x^2, x^3\}$

So the dimension is 4

- The set has 4 elements, so if these polynomials are linearly independent they form a basis

1 is not a multiple of zero

$2t$ is not a multiple of 1

$-2+4t^2$ is not a linear combination of 1, $2t$

$-12t+8t^3$ is not a linear combination of 1, $2t$ and $-2+4t^2$

$\rightarrow$ linear independent so the set is a basis.

14 homogeneous System $A\vec{x} = \vec{0}$

with A a 12×8 matrix

Two solutions $\vec{v}_1, \vec{v}_2$ to $A\vec{x} = 0$
that are linearly independent, and all
other solutions are linear combinations of
these 2 $\Rightarrow \{\vec{v}_1, \vec{v}_2\}$ is a basis for $Nul(A)$.

$\text{rank}(A) + 2 = 8 \Rightarrow \text{rank}(A) = 6$

6 equations could be enough since $Row(A)$
has a basis with 6 elements

16 $A\vec{x} = \vec{b}$ with A a 7×6 matrix

A can have 6 pivots since maximal rank of $A = 6$

$\Rightarrow$ if it does, for some target $\vec{b}$ $A\vec{x} = \vec{b}$ has a
unique solution.

When row reduced, A gets a zero row
(i.e. the equivalent matrix has a zero row.)
$\Rightarrow$ for some $\vec{b}$, there will be no solution to $A\vec{x} = \vec{b}$