Math 213 Section G1 Exam 2 (WITH SOLUTIONS)
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You may use a calculator and one 8”x11” two-sided sheet of formulas. No textbooks or lecture notes are allowed during the exam. PLEASE PRINT YOUR NAME AND YOUR NETID ON YOUR EXAM.

Problem 1.[15 points] Indicate the correct answer for each of the following questions. YOU DO NOT NEED TO SHOW WORK in this problem.

(a) Is it true that $C(n,0)+C(n,1)+C(n,2)+\cdots+C(n,n)=3^n$ where n is a positive integer?
Answer: No, it is false.

(b) Is it true that for an event E we have $p(E)+p(\overline{E})=1$?
Answer: Yes, it is true.

(c) Is it true that $a_n=5a_{n-1}-3a_{n-2}+1$ is a linear homogeneous recurrence relation?
Answer: No, it is false.

(d) What is the number of onto functions from a set with m elements to a set with n elements (where $m \geq n$)?
Answer:

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \cdots + (-1)^{n-1}C(n,n-1)1^m$$

Problem 2.[20 points] Find a particular sequence that satisfies the following recurrence relation:

$$a_n = a_{n-1} + 2a_{n-2} + 3 \cdot 2^n$$

Give all the details of your work.

Solution.

The characteristic equation of the associated homogeneous recurrence relation is

$$r^2 = r+2, \quad r^2 - r - 2 = 0, \quad (r-2)(r+1) = 0$$

with the roots $r_1 = 2, r_2 = -1$. The non-homogeneous part $F(n) = 3 \cdot 2^n$ is of the form $P_0(n)s^n$ where $P_0(n) = 3$ is a polynomial of degree zero and $s = 2$ is a root of the characteristic equation with multiplicity one. Therefore we can look for a particular solution to of our recurrence relation in the form

$$a_n = n^1 Q_0(n) s^n = nA2^n = An2^n$$

where A is some number.

We will determine A by substituting the above expression for a_n in the original recurrence relation:

$$a_n = An2^n, \quad a_{n-1} = A(n-1)2^{n-1}, \ a_{n-2} = A(n-2)2^{n-2}$$

and so

$$An2^{=}A(n-1)2^{n-1} + 2A(n-2)2^{n-2} + 3 \cdot 2^n.$$

Dividing by 2^{n-2} we get:

$$4An = 2An - 2A + 2An - 4A + 12 \quad \Rightarrow 6A = 12 \quad \rightarrow A = 2$$

Thus $a_n = 2n2^n = n2^{n+1}$ is a sequence satisfying the original recurrence relation.

Problem 3.[20 points] How many 4-card hands from a 52-card deck with exactly one red card are there? Give the details of your work.

Solution. There are 26 black card and 26 red cards in the deck. A 4-card hand with exactly one red card has one red card and three black cards. Therefore the number of ways to choose such a hand is

$$C(26,1) \cdot C(26,3) = 26 \cdot \frac{26 \cdot 25 \cdot 24}{1 \cdot 2 \cdot 3} = 67600$$

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Problem 4.[20 points] How many different strings can be made by permuting the letters of the word *ILLINOIS* if the two L's must be consecutive?

Give all the details of your work in this problem.

Solution.

Since the two L's must be consecutive, we can consider them as a single letter LL. Thus the word ILLINOIS contains $n_1 = 3$ letters I, $n_2 = 1$ letters LL, $n_3 = 1$ letters N, $n_4 = 1$ letters O and $n_5 = 1$ letters S. We have $n = n_1 + n_2 + n_3 + n_4 + n_5 = 3 + 1 + 1 + 1 + 1 = 7$.

Hence the number of permutations with two consecutive L's is

$$\frac{n!}{n_1!n_2!n_3!n_4!n_5!} = \frac{7!}{3!1!1!1!1!} = 840.$$

Problem 5.[25 points] How many integers n, such that $1 \leq n \leq 1000$, are not divisible by either of $5, 13, 31$?

Give all the details of your work.

Solution.

We have $N = 1000$ integers n such that $1 \leq n \leq 1000$. Let P_1 be the property that an integer is divisible by 5, let P_2 be the property that an integer is divisible by 13 and let P_3 be the property that an integer is divisible by 31. We have to find $N(P_1'P_2'P_3')$.

By the Inclusion-Exclusion Formula we have:

$$\begin{aligned} N(P_1'P_2'P_3') &= N - N(P_1) - N(P_2) - N(P_3) + \\ &+ N(P_1P_2) + N(P_1P_3) + N(P_2P_3) = \\ &= 1000 - \lfloor \frac{1000}{5} \rfloor - \lfloor \frac{1000}{13} \rfloor - \lfloor \frac{1000}{31} \rfloor + \\ &+ \lfloor \frac{1000}{5 \cdot 13} \rfloor + \lfloor \frac{1000}{5 \cdot 31} \rfloor + \lfloor \frac{1000}{13 \cdot 31} \rfloor - \lfloor \frac{1000}{5 \cdot 13 \cdot 31} \rfloor = \\ &= 1000 - 200 - 76 - 32 + 15 + 6 + 2 - 0 = 715 \end{aligned}$$