Math 213 Section G1 Exam 3

Prof. I.Kapovich December 1, 2000

You may use a calculator and one 8"x11" two-sided sheet of formulas. No textbooks or lecture notes are allowed during the exam. PLEASE PRINT YOUR NAME AND YOUR NETID ON YOUR EXAM.

Problem 1.[15 points] Indicate the correct answer for each of the following questions. YOU DO NOT NEED TO SHOW WORK in this problem.

(a) How many edges does a connected planar graph G with 3 vertices have if it divides the plane into 8 regions? **Answer:** 9.

(b) Is the graph W_n bipartite (where $n \geq 3$)? **Answer:** No.

(c) A connected graph has three vertices v_1, v_2, v_3 with degrees 7,9 and 10 accordingly. How many edges does this graph have? **Answer:** 13.

(d) Does the graph from part c) have an Euler path? **Answer:** Yes.

Problem 2.[20 points] Let $A = \{2, 6, 5, 9, 11\}$. Let R and S be relations on A defined as follows:

$$(a, b) \in R \text{ if and only if } a \in A, b \in A \text{ and } |a - b| \leq 2$$
$$(a, b) \in S \text{ if and only if } a \in A, b \in A \text{ and } gcd(a, b) > 1$$

(Here $gcd(a, b)$ means the greatest common divider of a and b). Let R' be a relation on the set of integers $\mathbb{Z}$ defined as $R' = \{(a, b) | a, b \in \mathbb{Z}, a^2 + b^2 > 10\}$.

For each of the relations R, S and R' determine if it is an equivalence relation and if yes, find its equivalence classes. Give all the details of your work.

Solution.

R is an equivalence relation with three equivalence classes: $[2] = \{2\}$, $[5] = [6] = \{5, 6\}$ and $[9] = [11] = \{9, 11\}$.

S is not transitive since $(2, 6) \in S$, $(6, 9) \in S$ but $(2, 9) \notin S$. Thus S is not an equivalence relation.

Finally R' is not reflexive. For example $1^2 + 1^2 = 2 < 10$ so that $(1, 1) \notin R'$. Thus R' is not an equivalence relation on $\mathbb{Z}$.

Problem 3.[20 points] Suppose a pseudo-graph G with vertices v_1, v_2, v_3 has adjacency matrix

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 4 \\ 1 & 4 & 0 \end{bmatrix}$$

How many paths of length three from v_1 to v_2 are there in G? Give the details of your work.

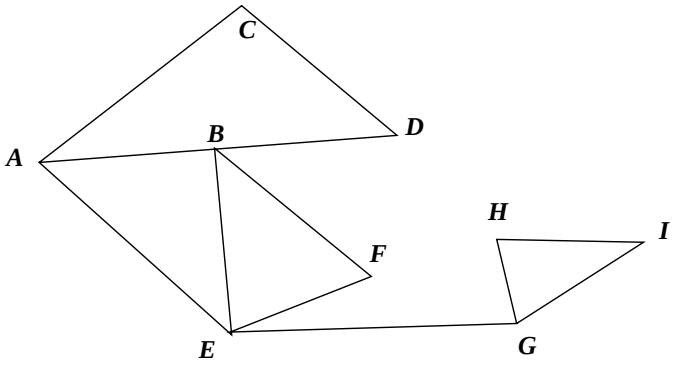
Solution.

We have

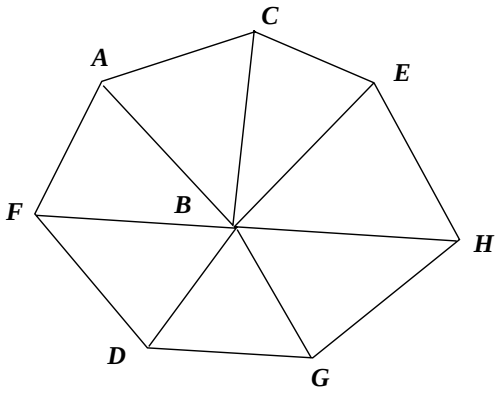
$$A^2 = \begin{bmatrix} 10 & 4 & 12 \\ 4 & 25 & 3 \\ 12 & 3 & 17 \end{bmatrix} \text{ and } A^3 = \begin{bmatrix} 24 & 78 & 26 \\ 78 & 24 & 104 \\ 26 & 104 & 24 \end{bmatrix}.$$

Hence there are 78 paths of length three from v_1 to v_2.

Problem 4.[20 points] For each of the following graphs determine if it has a Hamilton circuit. If not, explain why. If yes, find a Hamilton circuit.



A)



B)

Solution.

Graph A) does not have a Hamilton circuit since there is a cut-edge, namely edge EG.

Graph B) has a Hamilton circuit, for example

$$B, A, C, E, H, G, D, F, B.$$

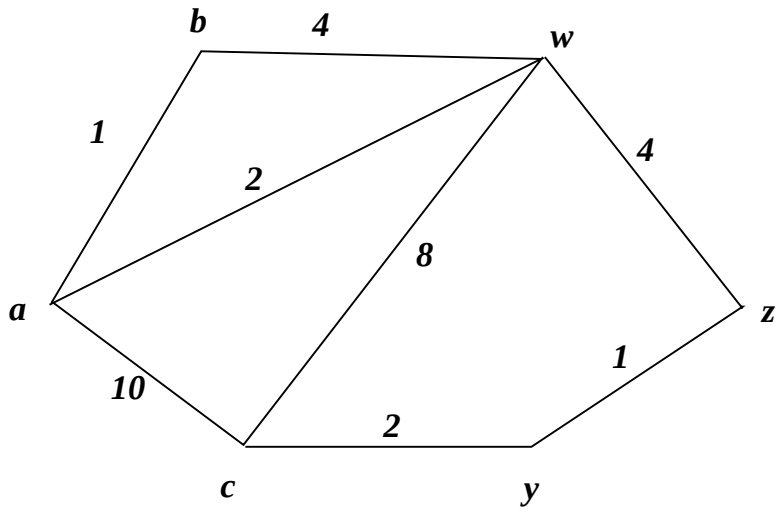
Problem 5.[25 points] Consider the weighted graph show in in the figure below.
(1) Apply Dijkstra's algorithm and find the lengths of shortest paths from the vertex a to each of the other vertices of the graph.
(2) Write down the shortest path from a to c.

Solution.

The lengths of the shortset paths are:

(1) from a to b: 1;
(2) from a to w: 2;
(3) from a to z: 6;
(4) from a to y: 7;
(5) from a to c: 9;

The shortest path from a to c is a, w, z, y, c.


b
4
w
1
2
4
8
a
z
1
10
2
c
y