Math 213 Section G1 Exam 1 (WITH SOLUTIONS)
Prof. I.Kapovich September 20, 2000

You may use a calculator and one 8"x11" two-sided sheet of formulas. No text-books or lecture notes are allowed during the exam. PLEASE PRINT YOUR NAME AND YOUR NETID ON YOUR EXAM.

Problem 1.[15 points] Indicate the correct answer for each of the following questions. YOU DO NOT NEED TO SHOW WORK in this problem.

Consider the set $A = \{3, \emptyset, -1, \{2, 5\}\}$

(a) What is the cardinality $|A|$ of A? **Answer:** $|A| = 4$
(b) What is the cardinality of the power set of A? **Answer:** $|P(A)| = 2^4 = 16$
(c) Is it true that $\{\emptyset, \{2, 5\}\} \subseteq A$? **Answer:** Yes
(d) Is it true that $5 \notin A$? **Answer:** Yes
(e) Find the set $A \cap \mathbb{N}$. **Answer:** $\{3\}$

Problem 2.[15 points] Indicate the correct answer for each of the following questions. YOU DO NOT NEED TO SHOW WORK in this problem.

Consider the function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined as $f(x) = x^3 - 5$ for each $x \in \mathbb{Z}$.

(a) Is the function f one-to-one? **Answer:** Yes
(b) Is it true that $f(x) = O(x^2)$? **Answer:** No
(c) Find $\sum_{i=1}^{2} \sum_{j=3}^{4} (f(i) + j)$. **Answer:** 12

Problem 3.[20 points] Show that among any 50 integers at least 9 will have the same reminder when divided by 6. Give the details of your work.

Solution.

There are only $k = 6$ possible reminders when an integer is devided by 6, namely the reminders $0, 1, 2, 3, 4, 5$. Therefore by the Generalized Pigenhole Principle if we take $N = 50$ integer numbers then at least one of our 6 reminders ("boxes") will have at least $\lceil \frac{N}{k} \rceil = \lceil \frac{50}{6} \rceil = \lceil 8.333 \rceil = 9$ numbers having this reminder when divided by 6.

Problem 4.[20 points]

Give all the details of your work in this problem.

(1) How many 4-letter words in the Latin alphabet begin with a vowel and end with a consonant?
(2) How many 4-letter words in the Latin alphabet either begin with a vowel or end with a consonant?

bf Solution.

(1) There are 26 letters in the alphabet, out of which 5 letters are voewls and 21 letters are consonants. Therefore by the Product Rule the number of 4-letter words which start with a voewl and end with a consonant is $5 \cdot 26 \cdot 26 \cdot 21 = 70,980$.

(2) Again, by the Product Rule there are $5 \cdot 26 \cdot 26 \cdot 26$ 4-letter words beginning with a vowel and there are $26 \cdot 26 \cdot 26 \cdot 21$ 4-letter words ending with a consonant. We have already showed in part (1) that there are $5 \cdot 26 \cdot 26 \cdot 21$ 4-letter word which start with a vowel and end with a consonant.

Therefore by the Inclusion-Exclusion Principle the number of words that either start with a vowel or end with a consonant is:

$$5 \cdot 26 \cdot 26 \cdot 26 + 26 \cdot 26 \cdot 26 \cdot 21 - 5 \cdot 26 \cdot 26 \cdot 21 = 385,996.$$

Problem 5.[30 points] Use induction to show that for every integer $n \geq 1$ we have:

$$(\dagger) \qquad \frac{1}{1^2} + \frac{1}{2^2} + \dots \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

Give all the details of your work.

Solution.

We will prove that inequality (†) holds for every $n \geq 1$ by induction on n.

(1) For $n = 1$ the left hand side of (†) is $\frac{1}{1^2} = 1$ and the right-hand side of (†) is $2 - \frac{1}{1} = 2 - 1 = 1$. Since $1 \leq 1$, the statement has been veryfied for $n = 1$.

(2) Suppose now that $n \geq 1$ and that (†) has been established for n. We need to show that it holds for $n + 1$, that is we need to prove

$$(*) \qquad \frac{1}{1^2} + \frac{1}{2^2} + \dots \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1}$$

Since we know that for n inequality (†) holds, we add to both sides of it $\frac{1}{(n+1)^2}$ and get

$$(**) \qquad \frac{1}{1^2} + \frac{1}{2^2} + \dots \frac{1}{n^2} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$$

If we can now show that

$$(***) \qquad 2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1},$$

this will imply that inequality $(**)$ holds as required.

We have:

$$\begin{aligned}
2 - \frac{1}{n} + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n+1} &\iff \\
-\frac{1}{n} + \frac{1}{(n+1)^2} \leq -\frac{1}{n+1} &\iff \\
\frac{1}{n} - \frac{1}{(n+1)^2} - \frac{1}{n+1} \geq 0 &\iff \\
\frac{(n+1)^2 - n - n(n+1)}{n(n+1)^2} \geq 0 &\iff \\
\frac{n^2 + 2n + 1 - n - n^2 - n}{n(n+1)^2} \geq 0 &\iff \\
\frac{1}{n(n+1)^2} \geq 0 &
\end{aligned}$$

The last inequality in this sequence is obviously true since $n \geq 1$.

Thus $(* * *)$ holds and therefore $(**)$ also holds, which completes the inductive step.